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This is the first volume of the proceedings of the Committee on the Undergraduate Program in Mathematics (CUPM) Geometry Conference, held at Santa Barbara in June, 1967. The purpose of the conference was to consider the status of geometry in colleges at the undergraduate level. The conference, attended by undergraduate mathematics teachers, involved lectures on various aspects of geometry, analysis of material presented, and an examination of the relevance of geometry to the undergraduate curriculum. In Part I of the proceedings are contained (1) an introduction by Walter Prenowitz and (2) the lectures on Convex Sets and the Combinatorial Theory of Convex Polytopes and Applications of Geometry, by Branko Grunbaum and Victor Klee. (RP)

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COMMITTEE ON THE UNDERGRADUATE PROGRAM IN MATHEMATICS

CUPM REPORT

AUGUST 1967

NUMBER 16

CUPM GEOMETRY CONFERENCE

PROCEEDINGS

PART I: CONVEXITY AND APPLICATIONS

Lectures by Branko Grünbaum and Victor Klee

U.S. DEPARTMENT OF HEALTH, EDUCATION & WELFARE
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MATHEMATICAL ASSOCIATION OF AMERICA

CUPM GEOMETRY CONFERENCE

Santa Barbara, California

June 12 - June 30, 1967

PROCEEDINGS OF THE CONFERENCE

Edited by Lincoln K. Durst

PART I: CONVEXITY AND APPLICATIONS

Lectures by Branko Grünbaum and Victor Klee

COMMITTEE ON THE UNDERGRADUATE PROGRAM IN MATHEMATICS

Mathematical Association of America

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FOREWORD

The Proceedings of the CUPM Geometry Conference will be issued in three parts:

I Convexity and Applications (Lectures by Grünbaum and Klee).

II Geometry in Other Subjects (Lectures by Gleason and Steenrod).

III Geometric Transformation Groups, and Other Topics (Lectures by Coxeter, and others).

The texts printed here are based on recordings made of the lectures and the discussions, and were prepared for publication by the assistants (Hausner, Reay, and Yale). The lecturers themselves were able to make minor changes and corrections on the final sheets, but an early deadline prevented major revision or extensive polishing of the texts. The typing for offset was done by Mrs. K. Black and the figures were prepared by Mr. David M. Youngdahl.

Lincoln K. Durst
Executive Director, CUPM

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Branko Grünbaum

INTRODUCTION

A Geometry Conference, sponsored by the Committee on the Undergraduate Program in Mathematics of the Mathematical Association of America, was held on the Santa Barbara campus of the University of California from June 12 to June 30, 1967.

The conference had its genesis in a meeting of geometers which was called by P. C. Hammer to consider the status of geometry in our schools and colleges. The meeting was held in Chicago on January 27, 1966 and was attended by more than twenty mathematicians including R. D. Anderson, Chairman of the Committee on the Undergraduate Program in Mathematics, and E. G. Begle, Director of the School Mathematics Study Group. It was generally agreed that the immediate focus of the problem of geometry lay at the undergraduate level (although its solution was related to questions involving the graduate curriculum and pre-college mathematics) and that the problem should be referred to CUPM. In May, CUPM authorized a small conference to consider the question further. A meeting was held in the fall of 1966 at UCLA, attended by S. S. Chern, L. K. Durst, P. C. Hammer, P. J. Kelly, V. L. Klee, Jr., W. Prenowitz, N. E. Steenrod, with Anderson as chairman. After preliminary discussion of the problem, Chern proposed a summer conference of undergraduate mathematics teachers who would attend lectures on various aspects of geometry, analyze the material presented, and examine its relevance to the undergraduate curriculum. Steenrod suggested that in addition lectures be given on the geometric underpinning of other branches of mathematics. This new and experimental format, which would involve features of a summer institute and a seminar on curriculum, quickly received approval of the group and of CUPM itself. Anderson appointed a Planning Committee consist-of Hammer, Kelly and Klee, with Prenowitz as chairman.

The Planning Committee proposed a conference of four weeks duration but consented to a reduction to three weeks for fiscal and other reasons, despite misgivings that there might not be enough time for a natural development of curriculum discussion. Twenty-one college teachers were chosen as participants by invitation, since the experimental nature of the conference did not require a large membership.

The following mathematicians lectured for a period of one or more weeks: H. S. M. Coxeter, A. M. Gleason, B. Grünbaum, V. L. Klee, Jr., N. E. Steenrod. Shorter series of lectures were given by H. Busemann, G. Culler, P. C. Hammer, P. J. Kelly and W. Prenowitz. Each lecture was followed by a discussion period--this helped to contribute a freshness of spirit to the discussion since questions, remarks and challenges did not have a chance to be forgotten or lose their cutting edge. Several discussions on curriculum were scheduled as the need arose. The program was supplemented by the showing of several films produced by the College Geometry Project of the University of Minnesota.

An important innovation was the selection of three assistants to the lecturers, younger mathematicians who were responsible for writing up the lecture notes and leading discussions on the material and its relation to the undergraduate curriculum. M. Hausner, J. R. Reay, and P. B. Yale were chosen for these assignments and carried them out with singular dedication.

The text which follows gives a record, sometimes in summary form, of the lectures and discussions of the conference.

Walter Prenowitz

LECTURE TOPICS

Herbert Busemann

The Simultaneous Approximation of n Real Numbers by Rationals.

An Application of Integral Geometry to the Calculus of Variations.

H. S. M. Coxeter

Transformation Groups from the Geometric Viewpoint.

Glen J. Culler

Some Computational Illustrations of Geometrical Properties in Functional Iteration.

Andrew M. Gleason

Geometry in Other Subjects.

Branko Grünbaum

Convex Sets and the Combinatorial Theory of Convex Polytopes.

Preston C. Hammer

Generalizations in Geometry.

Paul J. Kelly

The Nature and Importance of Elementary Geometry in a Modern Education.

Victor Klee

Applications of Geometry

Walter Prenowitz

Joining and Extending as Geometric Operations: A Coordinate-Free Approach to n-space.

Norman E. Steenrod

Geometry in Other Subjects.

MEMBERS OF THE CONFERENCE

Russell V. Benson
California State College, Fullerton

Gavin Bjork
Portland State College

John W. Blattner
San Fernando Valley State College

Herbert Busemann (Lecturer)
University of Southern California

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University of California,
Santa Barbara

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University of California, Davis

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University of Washington

Rev. John E. Koehler
Seattle University

Sister M. Justin Markham
St. Joseph College

Michael H. Millar
Stanford University

H. Stewart Moredock
Sacramento State College

Richard B. Paine
Colorado College

Walter Prenowitz (Chairman)
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Paul B. Yale (Assistant)
Pomona College

PROCEEDINGS

APPLICATIONS OF GEOMETRY

Lectures by Victor Klee

(Lecture notes by Melvin Hausner and John Reay)

Lecture I.

During the past few years, we have witnessed the effects of the many efforts to revise and improve the mathematical curriculum (SMSG, CUPM, etc.). These efforts have proved to be widely influential and beneficial--much more so than we might have imagined when they were initiated. However, the possibility of further progress should not be ignored. In particular, an informal session at the 1966 Chicago A.M.S. meeting made it clear that many geometers favor a broad further revision of the geometry curriculum from the kindergarten through the senior undergraduate level. Since such a broad attack was clearly not feasible, it was decided to start at the college level. Hence we are gathered here.

Let us consider the following four questions.

1. Is collegiate geometry instruction in a state of decline?
2. If so, is this undesirable?
3. If this is undesirable, what should be done to improve the situation?
4. What should a college geometry course consist of?

Our assumption is that questions 1 and 2 have affirmative answers and that questions 3 and 4 are only slightly different. Once question 4 has been answered, the next step will be to write the appropriate books.

(It is profitable to compare the situation of geometry with that of algebra. In algebra, thousands of students have learned from the books by van der Waerden or from Birkhoff and MacLane or their imitators. Hence the algebraists have a common experience which serves to strengthen their subject.

I do not believe the universality of these books was a result of general agreement on what belonged in an algebra course at the time they were written. Rather, it occurred because the books were written so well that their readers came to think that this was the only natural way to proceed.)

What are the desirable characteristics of an undergraduate collegiate geometry course? In my opinion, they should include the following points. Except for the first and third, these apply as well to any college mathematics course.

a) The course should be n-dimensional, and even infinite-dimensional if this is possible with little extra cost. Anything less will be a letdown for the student and will minimize the usefulness of the course.

b) There should be a unifying theme throughout the course. For example, it might be the study of certain objects, such as differentiable manifolds or convex bodies. Or it might be some notion, such as invariance or symmetry.

c) It is more important to study the geometric objects, their structure and their properties, than to have an esthetically pleasing axiom scheme. One should use the most powerful approach rather than the most esthetic one. Thus the axiomatic basis should be specifically "geometrical" in nature only if no loss of efficiency results from this.

d) It should emphasize the points of contact with other areas of mathematics.

e) It should include applications to science and technology wherever possible.

f) Unsolved problems should be mentioned in order to whet the students' interest.

Subjects which might satisfy these criteria for a geometry course are differentiable manifolds, algebraic topology, n-dimensional projective geometry, and the geometry of convex bodies. There are of course several other possibilities.

My candidate is a course on convex bodies. As far as point a) is concerned, the subject is easily carried out in n dimensions. As Grünbaum pointed out, many of the methods used for n-space are essentially two- or three-dimensional, making motivation and intuition very easy in n dimensions even for the inexperienced. As to point b), the unifying theme is the study of the properties of convex sets. This is very pictorial, and close in spirit to the familiar Euclidean geometry, but it makes much more contact with modern mathematics. The method which seems to fit the criterion of point c) is to use real vector spaces as the setting for the theory. However, if Prenowitz's "join geometry" approach can be developed in more detail, it might very well be used instead. The possibilities for point f) are excellent (cf. Grünbaum's lectures). Finally there are many connections of convexity with other areas of mathematics, and many applications in science and technology. Let us mention just a few.

1. In functional analysis, three of the most important tools are separation theorems for convex sets (the basic separation theorem being equivalent to the Hahn-Banach theorem), extreme-point theorems for convex sets, and fixed-point theorems for convex sets. Inclusion of these topics in a convexity course serves simultaneously to teach the student some interesting geometrical facts and to prepare him for a later course in functional analysis.

2. In a very natural way, a number of topological notions can be brought into a course on convexity. For example, the Euler characteristic can be approached in a combinatorial way based on convex sets. Having defined simplicial subdivisions and proved that a simplex admits subdivisions of

arbitrarily small mesh, then by using Ky Fan's combinatorial lemmas (taking perhaps one hour), it is possible to prove (perhaps in two hours) such basic results as Brouwer's fixed point theorem, the Borsuk-Ulam antipodal mapping theorem, the invariance of domain theorem, the Lebesgue tiling theorem, etc. One can then proceed to use these results in obtaining further properties of convex sets.

3. In a study of the combinatorial properties of convex sets, various combinatorial identities enter in a natural way and of course there are many contacts with graph theory. At present, matroids do not seem to arise naturally in such a course, but techniques now being developed will probably change this situation in a few years. (Jon Folkman has a new combinatorial notion which is related to positive bases as matroids are related to linear bases.)

4. In inequality theory, many results are best viewed from the point of view of convex functions. There are applications of convexity in summability theory and in many other parts of mathematics.

There are many areas of modern applied mathematics which use convexity in a significant way. Some of them are listed below.

5. In linear programming theory, the study of convex polytopes forms the essential geometric background.

6. In much of nonlinear programming theory, convex or concave functions form the background.

7. Control theory uses the separation theorems, as well as Liapounoff's theorem asserting the convexity of the range of a vector measure.

8. The theory of pattern classification uses separating hyperplanes in a fundamental way, and in particular Schläfli's theorem on the number of regions

determined by a finite set of hyperplanes in general position through the origin.

9. Parts of information theory are closely related to packing problems for convex bodies.

I readily confess that I have not attained a sufficiently high degree of organization to cover all the above topics in a one-year course, but I believe it to be possible. The unifying theme makes this less of a hodge-podge than it appears to be.

Discussion.

The question of a unifying theme was brought up. In particular, Klee admitted that this was not so apparent in Birkhoff and MacLane, although the notion of a group permeates this text. In this connection Klee felt that a course based on the notion of transformation as a unifying theme would be too bland in nature, with too few deep or exciting geometrical results.

Point c) of the ideal geometry course was discussed in some detail. This led to the general question of whether the course should be content-oriented or foundation-oriented, with Klee and most of the vocal participants choosing the former. However, when it was suggested that a mathematics course without axioms is simply not a mathematics course, Klee stressed that he did not wish to eliminate the axioms--he wanted to use an efficient axioms system and not concentrate on esthetics. In any event, the underlying axioms would be stated.

When the question of the importance of n-dimensional space was brought up, it was pointed out that a junior or senior level course was under discussion and that this should not cause problems at that level.

Finally, there was general agreement that a year's course on convex bodies should not be the sole upper level geometry course. It should be one of several. But Klee felt that one geometry course which has some depth is better than the customary hodge-podge course.

Lecture II.

We shall now discuss some applications of convexity which might be included in the sort of course Grünbaum is discussing. These are not applications in the strictest sense of the term. Rather, they illustrate the sort of "pure applied mathematics" in which one takes certain objects which come from applied mathematics and then one studies them for their own sake, just as one would any other mathematical object. (Note that much of mathematics has developed in exactly this way.)

We now introduce a few mathematical notions which form the necessary background for the material. These will be elaborated in Grünbaum's lectures.

Definition. An extreme point p of a convex set C is a point p of C such that $C \sim \{p\}$ is convex. (We use the notation $A \sim B$ for set-theoretic difference.) Equivalently, p is an extreme point of the convex set C if p is in C but is not the mid-point of any segment joining two points of C.

For example, the extreme points of a plane convex polygon are the vertices of that polygon.

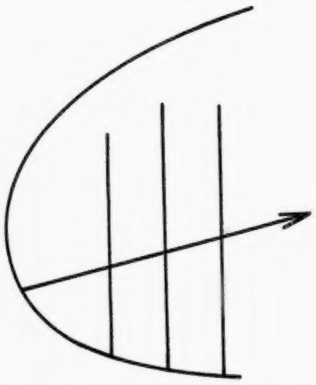
Extreme points and techniques for manipulating them constitute one of the three aspects of convex sets which are most important in connection with applications of convexity to other portions of mathematics. (The other two are separation theorems and fixed-point properties.) The importance of extreme points is due in part to the following fact.

Theorem. If K is compact, convex, and finite dimensional, then K is the convex hull of its extreme points; that is, $K = \text{con ext } K$.

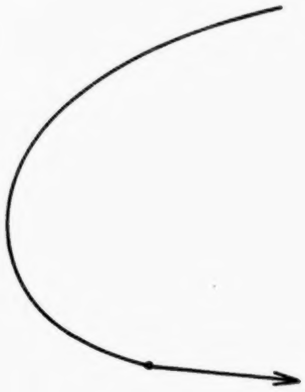
(This result goes back to Minkowski. In the corresponding infinite-dimensional result of Krein and Milman it is necessary to take the closed convex hull of the extreme points.)

If we abandon the restriction of compactness, we have the following analogous result.

Theorem. If K is closed, convex, finite-dimensional, and line-free (no line is contained in K), then K is the convex hull of its extreme points and its extreme rays; that is $K = \mathrm{con}(\mathrm{ext}\, K \cup \mathrm{rex}\, K)$. (An extreme ray of a convex set K is a half line contained in K which is not crossed by any segment in K.)



A ray which is not extreme



An extreme ray

Another important reason for studying extreme points is in their application to a wide variety of practical optimization problems. For example, suppose one has a cost function which is to be minimized. In many cases, this function f is defined over a convex set C and in many applications f is affine, or at least concave. A basic property of such functions is as follows.

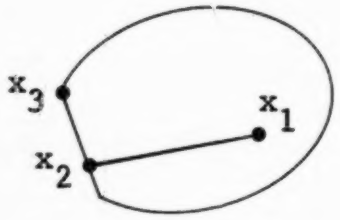
Theorem. If a concave function attains its minimum on a line-free closed convex set, it does so at an extreme point.

For the above result, as for the remainder of these lectures, we are assuming that the convex set is finite-dimensional.

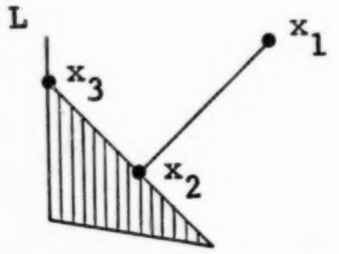
We would like to indicate a more or less constructive method of obtaining this minimizing extreme point, starting from an arbitrary minimizing point. Let us consider for the sake of simplicity a compact set, and let us first consider the simpler problem of finding an extreme point.

Grünbaum will probably describe a quick inductive proof of the existence of extreme points, but the following procedure is more constructive in nature.

In the diagram, we start at any point x_1. If it is not extreme, it is the mid-point of some segment. Extend this segment in one direction as far as possible, to a boundary point x_2. We continue the process with x_2, and in this way we obtain a sequence $x_1, x_2, \ldots$ of points. We claim that this procedure must terminate after at most $d+1$ steps, where d is the dimension of the set.



To see this, note that the point x_3 cannot be on the line x_1x_2, by the special choice of x_2. We claim similarly that x_4 cannot be in the plane $x_1x_2x_3$, and so on. The reason is indicated in the following figure. If it



were possible to continue along the line L in the plane $x_1x_2x_3$, then, since we are in a convex set, the shaded triangle would be in that set. This implies that x_1x_2 can be extended beyond x_2 which is a contradiction of the choice of x_2. Similar arguments hold for higher dimensions. Of course, the pictorial argument can be made formal. Now, if f is a given concave function, assumed continuous for the purposes of illustration, then a similar process may be applied to find its minimum. For if the minimum is attained at x_1, and if x_1 is not extreme, it is an immediate consequence of the notion of concavity that the minimum is also attained at one or the other endpoint of any segment through x_1, say at x_2. We may then continue as above to prove the stated theorem.

The required definitions of concavity and convexity, along with the pictorial idea are as follows.

Definition. Let C be a convex set, and let f be a real-valued function on C. Then

1) f is concave if $f(\lambda x + (1-\lambda)y) \geqq \lambda f(x) + (1-\lambda)f(y)$ for all x and y in C and all λ between 0 and 1:

2) f is convex if the above inequality is reversed. Equivalently, f is convex if -f is concave:

3) f is affine if it is both convex and concave; the above inequality is replaced by an equality:

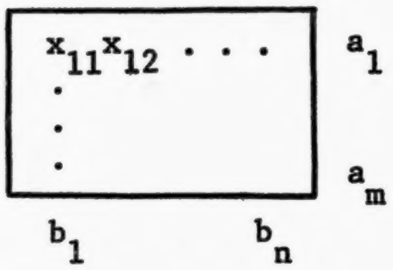
These formulations translate the usual geometric notions. Thus, f is concave, convex, or affine if the graph lies respectively above, below, or on any chord.

It is clear why the minimum of a concave function on a line segment must be attained at an endpoint. For if not, we would have a graph with points:

which is clearly not possible for a concave function.

We now consider a specific class of minimization problems, namely, the transportation problems. Such a problem (an $m \times n$ transportation problem) is specified by a positive m-vector $a = (a_1,\ldots,a_m)$, $a_i > 0$, and a positive n-vector $b = (b_1,\ldots,b_n)$, $b_i > 0$, where $\Sigma_{i=1}^m a_i = \Sigma_{j=1}^n b_j$. We suppose that we have a homogeneous, infinitely divisible commodity located at the sources

$A_1, \ldots, A_m$. The amount a_i of the commodity is at the source A_i. The problem is to transport this commodity to the destinations (or sinks) B_j in such a way that the total amount b_j of the quantity arrives at B_j. A solution of this problem is an $m \times n$ matrix $x = (x_{ij})$, where x_{ij} is the amount of the material from source A_i which is sent to the sink B_j. Thus, the solution x is a matrix which satisfies the conditions:



1. $x_{ij} \geqq 0$
2. $\Sigma_{j=1}^{n} x_{ij} = a_i$ (row sum)
3. $\Sigma_{i=1}^{m} x_{ij} = b_j$ (column sum)

The rectangle is a convenient pictorial device to help us remember the size of x and the row and column conditions. The solutions, as a set of $m \times n$ matrices, form a subset of E^{mn}. Furthermore, the defining equations and inequalities for x show that the set of such x's is a convex polytope. To see this, note that each of the mn inequalities $x_{ij} \geqq 0$ defines a closed halfspace, and each of the equalities determines a hyperplane which is the intersection of the two closed halfspaces determined by that hyperplane. Thus the set of matrices x satisfying the required conditions is an intersection of finitely many closed halfspaces, namely a polyhedron. It is clearly bounded, since each component is bounded by $\Sigma_i a_i$. Thus it is a polytope.

Definition. If a and b are vectors as above, the set of matrices x satisfying the above conditions is called the transportation polytope $T(a,b)$.

Now suppose there are cost functions c_{ij}, where $c_{ij}(\alpha)$ is the cost of sending an amount α from source A_i to sink B_j. Then the cost of a particular transportation scheme x is simply

$$C(x) = \Sigma\, c_{ij}(x_{ij}).$$

In most practical problems the functions c_{ij} will be concave, whence C is also concave. When the c_{ij}'s are linear this is an example of a problem in linear programming, namely to minimize a linear function over a convex polyhedron. We know that the minimum is attained at some vertex of the polytope. (A vertex is an extreme point of a polyhedron.) It is also easy to see that the set of minimizing points is the convex hull of the set of minimizing vertices. The subject of linear programming involves various algorithms to find these minimal vertices. We should like to point out that sometimes this is not a simple problem even though there are only finitely many vertices. The number of vertices may be very large, so that a direct search is impractical. Further, the polyhedron is generally not given in terms of its vertices but only as the solution set of a system of linear inequalities.

It is not the purpose of these lectures to discuss practical methods of solving linear programming problems. However we can sketch the practical technique usually employed. First find one vertex. (Often this is not trivial.) There are computationally practical ways whereby, having a vertex of the polyhedron, one can "look" at all of the adjacent vertices. Of the adjacent vertices, choose the one which gives the smallest value to the cost function or which in some other sense represents a maximum improvement. Then continue the process. It must terminate at a minimizing vertex, as follows from a simple theorem in the geometry of polytopes. This is all for linear cost functions. For concave functions the situation is much more complicated.

Although linear programming problems are concerned with finding an optimizing vertex of the feasible region, there are closely related problems in which one may want to know all of the vertices. An example would be a transportation problem in which an enemy is choosing the non-negative coefficients c_{ij} of the linear cost function (subject to $\Sigma c_{ij} = 1$) in the hope of maximizing $C(x)$.

We now study the geometry of $T(a,b)$. We precede this sequence of results by the remark that the subject itself is not of overwhelming mathematical interest. It is given simply as a sample of what could be done, in a course treating convex polytopes, to increase the students' contact with the more applied aspects of the subject.

Theorem 1. $\dim T(a,b) = (m-1)(n-1)$

We prove this result in several steps.

a) The equality constraints (the row and column conditions) define a flat of dimension $(m-1)(n-1)$. Each of the row and column sum conditions defines a hyperplane. The claim is that the intersection of these hyperplanes has the required dimension. Momentarily, the inequality constraints are being ignored.

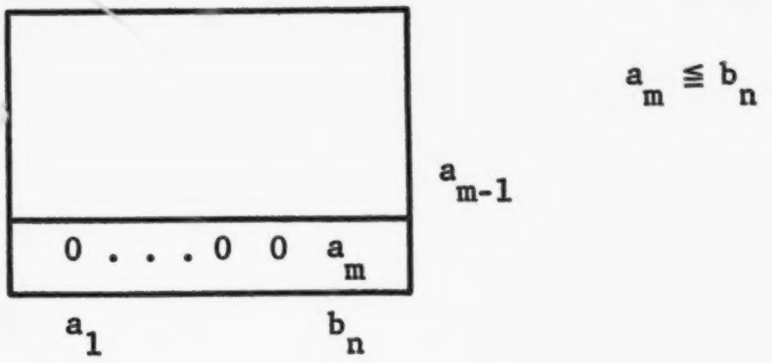
To prove a), we note that there are $m+n$ linear functions here. Let, for example, $\rho_i(x) = \Sigma_j x_{ij}$ (= $i^{\underline{th}}$ row sum) and $\gamma_j(x) = \Sigma_i x_{ij}$ (= $j^{\underline{th}}$ column sum). These $m+n$ linear functions are not linearly independent, since $\Sigma_i \rho_i = \Sigma_j \gamma_j$. However, if we delete γ_n, the remaining $m + n - 1$ linear functionals are linearly independent. (For example, we may easily arrange to make any one of these equal to 1, while the others equal 0.) Thus the rank of this system of linear functions equals $m + n - 1$ and the dimension of the flat in question is $mn - (m+n-1) = (m-1)(n-1)$.

(Remark. Using the theorem of Carathéodory, this result shows that each feasible transportation scheme (point of $T(a,b)$) is a convex combination of $(m-1)(n-1) + 1$ or fewer vertices. For $n \times n$ doubly stochastic matrices, the vertices may be identified as the permutation matrices. Thus, every

$n \times n$ doubly stochastic matrix is a convex combination of $n^2 - 2n + 2$ or fewer permutation matrices.)

b) For each i and j, there exists a vertex x of $T(a,b)$ such that $x_{ij} > 0$.

With no loss ~~in~~ generality, we may assume $i = m$, and $j = n$. Then we use the "southeast corner rule," $x_{mn} = \min(a_m, b_n)$ and proceed at each stage to fill in the southeast corner



with as large an entry as possible; that is

$$x_{ij} = \min(a_i - \Sigma_{j<c\leqq n}\, x_{ic},\; b_j - \Sigma_{i<r\leqq m}\, x_{rj}).$$

Then we claim that the matrix so constructed is a vertex. The proof is by an easy induction which is left to the reader.

c) There is a point of $T(a,b)$ which is interior to the positive orthant determined by the inequalities $x_{ij} > 0$.

To see this, let p_{ij} be the mn vertices determined in part b). Now take their average, $p = (\Sigma_{i,j}\, p_{ij})/mn$. Then p is a convex combination of vertices, hence an element of the polytope. But p clearly has all of its entries positive and hence is in the interior of the positive orthant.

Finally, the proof of Theorem 1 follows immediately from a) and c), since $T(a,b)$ is the intersection of a flat of dimension $(m-1)(n-1)$ which passes through an interior point of the positive orthant.

Discussion.

The discussion clarified the final part of the proof of Theorem 1. In addition, the formal definitions of convex and concave functions were presented, as given in the notes. The importance of these functions were stressed by Klee, insofar as practical maximum and minimum problems were concerned. For example, the minimum of a concave function is always attained at an extreme point, while for convex functions, a local minimum is a global minimum. Problems of minimizing or maximizing nonconvex and nonconcave functions tend to be much more difficult.

Klee pointed out that transportation problems are a real application, not a contrived one. In particular, a significant portion of present day computer time is devoted to the solution of linear programming problems.

The question of computing volumes of polytopes was brought up as a practical one in geometric probability. The difficulty even for dimensions 2 and 3 was indicated. Two-dimensional problems can be done with some ingenuity, and higher dimensional problems with a large amount of symmetry can also be handled. But otherwise, nobody can do it. Klamkin pointed out that Pólya, in his thesis, considered the problem of finding the volume cut off by two parallel hyperplanes in a cube. Explicit formulas were given and were derived by Laplace transforms.

Lecture III.

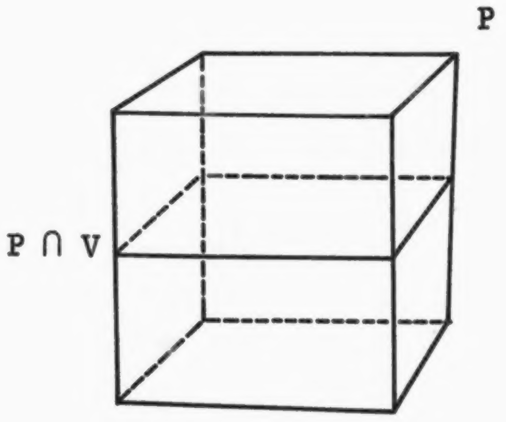
We continue the discussion of the properties of transportation polytopes with a theorem about the number of facets of $T(a,b)$. (A facet of a d-dimensional polytope is a face of dimension $d-1$. In the study of polytopes, the faces of extreme dimension (namely the facets and the vertices) are usually the most interesting faces.) The theorem is not of any great importance as far as applications are concerned, but in proving it we will learn things about the structure of the problem which are of interest for the applications. Assume from now on that $m \leqq n$. This is really not an essential limitation, since mathematically there is no distinction between sources and sinks.

Theorem 2. $T(a,b)$ has exactly one facet when $m = 1$ and two facets when $m = n = 2$. The number of facets is otherwise between $(m-1)n$ and mn, each possibility being realized by some transportation polytope.

Proof. The first two cases are trivial. We prove a number of statements, each of which is easy by itself.

a) Each facet of $T(a,b)$ is of the form $\{x \in T(a,b) \mid$ a particular coordinate $x_{ij} = 0\}$. (This does not assert that each set of this form is a facet. Note that this does give the upper bound on the number of facets.)

$T(a,b)$ is the intersection of a flat with the positive orthant, but this orthant itself has exactly mn facets. Thus it suffices to prove that when P is a polyhedron, V is a flat, and F is a facet of $P \cap V$ then F is the intersection of V with some facet of P.



(For example if P is a cube, V a plane cutting its middle, the facets of $P \cap V$ are the four edges of the (middle) square.) The proof uses a characterization of facets F of a polyhedron as maximal convex subsets of the boundary of the polyhedron.

b) A set of the form in a) is a facet if and only if its dimension is $(m-1)(n-1) - 1$.

This is evident, since each set of this form is the intersection of the polytope (of dimension $(m-1)(n-1)$) with a supporting hyperplane.

Thus we want to decide for which pairs (i,j) does this form give a facet of the polytope.

c) Let $\sigma = \Sigma_{r=1}^{m} a_r = \Sigma_{c=1}^{n} b_c$, and fix (i,j). Note that σ is the total amount of the commodity shipped, while $a_i + b_j$ is the sum of the amount shipped from source A_i and the amount received by the sink B_j.

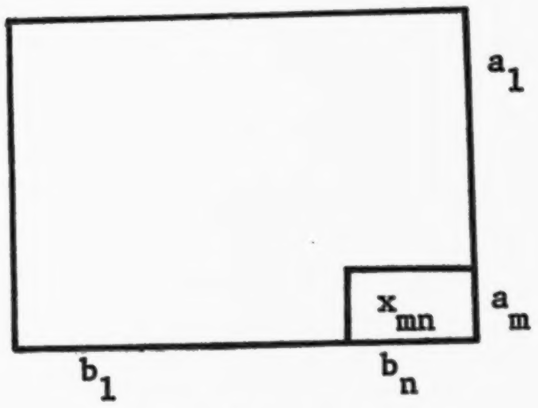
$\sigma < a_i + b_j$ iff $\{x \in T(a,b) \mid x_{ij} = 0\}$ misses $T(a,b)$

$\sigma = a_i + b_j$ iff $\{x | x_{ij} = 0\}$ determines a vertex of $T(a,b)$,

$\sigma > a_i + b_j$ iff $\{x | x_{ij} = 0\}$ determines a facet of $T(a,b)$.

This gives a relation between the positive orthant and the flat determined by the set of a). In general the intersection of a supporting hyperplane with a polytope may be a facet or a vertex, or a face of some intermediate dimension. But here the intermediate possibilities cannot occur.

Rather than prove this, we consider a picture of the situation. Each point of $T(a,b)$ is a matrix x, as shown. Assume (for notational simplicity, and without loss of generality) that $(i,j) = (m,n)$. The question is: What can we say about matrices x in $T(a,b)$ with $x_{mn} = 0$? If $a_m + b_n > \sigma$, i.e., $b_n > \sigma - a_m = \Sigma_{i=1}^{m-1} a_i$, there can be no non-negative x in $T(a,b)$ for which $x_{mn} = 0$, for any such x would give $b_n = \Sigma_{i=1}^{m-1} x_{in} \leq \Sigma_{i=1}^{m-1} a_i$. If



$a_m + b_n = \sigma$ it follows that all elements in the last row and column must be as large as possible. Thus the rest of the matrix must be zero, i.e., the hyperplane meets $T(a,b)$ at exactly one point, so this point is a vertex of the polytope. If $a_m + b_n < \sigma$, the entries in the last row and column may be chosen less than the maximum amount (e.g., $x_{1n} < a_1$, etc.), and are subject only to the two restrictions $\Sigma_{i=1}^{m} x_{in} = b_n$ and $\Sigma_{i=1}^{n} x_{mi} = a_m$. Thus (within restrictions) we have a "free choice" of $(m-2)(n-2)$ elements of the reduced matrix and $(m-2)$ and $(n-2)$ free choices in the last row and column, i.e., $(m-2)(n-2) + (m-2) + (n-2) = (m-1)(n-1) - 1$ free choices. Thus $\{x | x_{mn} = 0\}$ determines a facet of $T(a,b)$.

Note that this result is of interest because if an opponent is choosing the cost function c_{ij}, he may make use of the following fact: if $a_i + b_j > \sigma$ then clearly every feasible scheme must send something from source A_i to sink B_j.

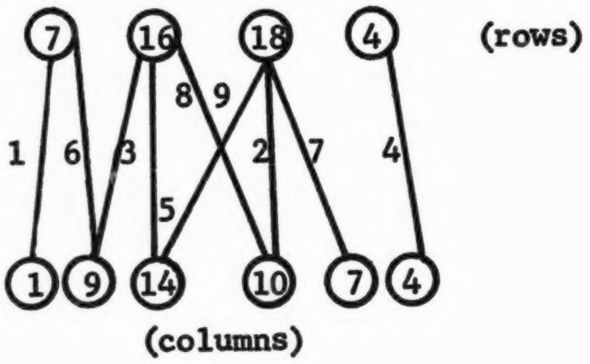
d) If $\sigma \leqq a_r + b_s$ and $\sigma \leqq a_u + b_v$ with $r \neq u$, $s \neq v$, then $m = n = 2$. That is, if we have two critical points in distinct rows and columns of x, then $m = n = 2$.

This is clear since $a_r \geqq \sigma - b_s = \Sigma_{j \neq s} b_j \geqq b_v \geqq \sigma - a_u = \Sigma_{i \neq u} a_i \geqq a_r$, with the second and fourth "$\geqq$" being "$=$" only if respectively $n = 2$ and $m = 2$. Thus all these "bad points" (i,j) must be in the same row or in the same column, and there can be at most n of them (since $m \leqq n$). But these points (i,j) are precisely the places in the $m \times n$ matrix where $\{x \in T(a,b) \mid x_{ij} = 0\}$ is not a facet, by part c). Thus at least $mn - n = (m-1)n$ of these sets must be facets of $T(a,b)$. This proves the theorem except for showing that all possibilities between $(m-1)n$ and mn are actually realized.

We now turn to the number of vertices of $T(a,b)$. We first characterize the vertices in a useful way. This characterization will not be explicit enough to allow us to count them in general. Leading up to this, I would like to mention a standard correspondence between matrices and bipartite graphs which will be useful in some of the counting processes.

Associated with any $m \times n$ matrix we have an undirected graph whose nodes or vertices are divided into two sets; one set of nodes corresponding to the rows of the matrix, the other set of nodes corresponding to the columns of the matrix. We connect the $i^{\underline{th}}$ row node to the $j^{\underline{th}}$ column node if and only if the corresponding entry in the matrix is non-zero. For example, this matrix and graph correspond.

$$\begin{bmatrix} 1 & 6 & 0 & 0 & 0 & 0 \\ 0 & 3 & 5 & 8 & 0 & 0 \\ 0 & 0 & 9 & 2 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$



(numbers in each node are
row or column sums)

These graphs are associated with the transportation problem in a natural way, since we can think of the row nodes as sources and the column nodes as sinks. The arcs may then be labeled with the amount of shipment along that arc. We will prove that the feasible transportation schemes (i.e., points $x \in T(a,b)$) which are vertices of $T(a,b)$ are exactly those whose graphs are trees or "forests" (unions of a finite set of disjoint trees).

Definition. A loop for the matrix x is a sequence $(i_1,j_1)(i_1,j_2)(i_2,j_2)\cdots(i_k,j_1)$ where we successively change only one of the two numbers i or j, such that

a) there are no repetitions, and

b) the matrix has non-zero entries in all the indicated places.

Equivalently, a matrix has a loop if and only if its graph has a circuit, since the only way you can leave a "row" node is to go to a "column" node in a bipartite graph, and to leave a column node you must go to a row node. Thus the following theorem has a geometric as well as graph-theoretic interest.

Theorem 3. The vertices of the polytope $T(a,b)$ are exactly those $x \in T(a,b)$ such that the matrix x admits no loop.

Proof. Suppose $x \in T(a,b)$ and x has a loop $(i_1,j_1)(i_2,j_1)\cdots(i_s,j_s)$ (i_1,j_s). Set $2\epsilon = \min \{x_{ij} \mid (i,j) \text{ listed}\} > 0$. Define a matrix y by

$$y_{ij} = x_{ij} \quad \text{if} \quad (i,j) \text{ is not listed in the loop,}$$
$$y_{ij} = x_{ij} \pm \epsilon \text{ if } \quad (i,j) \text{ is in the loop}$$

where ϵ is alternately added and subtracted as we go around the loop. This has the effect of leaving all row sums and all column sums fixed and hence y, $\frac{1}{2}(x+y)$, and $\frac{1}{2}(x-y)$ are all members of the convex polytope $T(a,b)$. But $x = \frac{1}{2}(x+y) + \frac{1}{2}(x-y)$ and thus x is not an extreme point.

Conversely, if $x \in T(a,b) \sim \text{ext}\, T(a,b)$ we will show that the matrix x has a loop. If x is not a vertex of $T(a,b)$ then there is some y such that $x \pm y \in T(a,b)$, and the row sums and column sums of y must necessarily all be zero. Now choose $y_{i_1 j_1} \neq 0$. Since all row and column sums of y are zero we can continue to choose pairs (i,j) which define a loop in y for which the corresponding y_{ij} are non-zero. Since $x+y$ and $x-y$, being in $T(a,b)$ must have all non-negative entries, it follows that each x_{ij} must be positive and we thus have a loop in matrix x. This completes the proof.

Theorem 4. For any matrix, the following are equivalent:

a) x admits no loop

b) $x \in T(a,b)$ is a vertex of $T(a,b)$

c) every submatrix of x (obtained by deleting certain rows and columns) admits a distinguished row or column. (Definition: A row or column is distinguished if it has at most one non-zero element.)

d) any $r \times c$ submatrix has at least $r - c + 1$ distinguished rows when $r \geqq c$ (distinguished columns when $c \geqq r$).

e) every square submatrix has a distinguished row or a distinguished column

f) every square submatrix has both a distinguished row and a distinguished column

g) every square submatrix of order k has at most $2k - 1$ non-zero entries.

The conditions of this theorem could, of course, be put into graph theoretic terms. For example, c) implies that the bipartite graph of the matrix x must have at least one "dead-end." To say a certain row is distinguished corresponds to saying that a particular row node is joined to at most one other node. Condition e) asserts that if we pick any set of nodes containing the same number of row nodes as column nodes, and the arcs which join these chosen nodes, then there must be at least one dead-end in that subgraph. Condition f) asserts that there is a dead-end row node and a dead-end column node in each such subgraph.

None of these conditions allows us to count the vertices of a really complicated transportation polytope, although they do allow us to look at certain points $x \in T(a,b)$ and say immediately that they are not vertices.

Discussion.

Hausner pointed out that there are only finitely many combinatorial types of the polytope $T(a,b) \subset E^{mn}$ and yet the combinatorial structure depends upon the continuous parameter $(a,b) \in E^{m+n}$. He asked what determined the boundary of a set of points $(a,b) \in E^{m+n}$ which gave a transportation polytope $T(a,b) \subset E^{mn}$ of a particular combinatorial type. (Definition: Polytopes P and P' are of the same combinatorial type if the lattices $\mathfrak{F}(P)$ and $\mathfrak{F}(P')$ of their faces are isomorphic.) Klee answered that this was typical of a class of similar questions that could be asked, most of which are hard. Perhaps you could prove that two transportation polytopes in E^{mn} had the same combinatorial type provided a certain characteristic function $\chi(a,b)$ assumed the same value on the pairs (a,b) and (a',b') which defined the given transportation polytopes. The characteristic function χ would be defined in terms of the class of all subsets of the set of $m + n$ coordinates of (a,b).

Certain game theory results have applications to economics or business, but no one suggested any peaceful applications of the game-theoretic transportation problem in which an enemy is determining the cost function. Klee remarked that since all "bad points" of a matrix x (i.e., pairs (i,j) for which $\sigma < a_i + b_j$ in the notation of Theorem 2) must occur on one row or one column, there will be one critical source or sink in these cases whose lines of communication must be closely guarded.

Lecture IV.

We will continue our discussion of transportation polytopes, but will omit the details of some of the counting arguments. These would be similar in spirit to those done above, but considerably more complicated and time-consuming.

We first consider the possible number of vertices. A transportation problem determined by $(a,b) \subset E^{m+n}$ is called *degenerate* if the sum of the elements of some proper subset of $\{a_i \mid 1 \leqq i \leqq m\}$ is equal to the sum of some proper subset of $\{b_j \mid 1 \leqq j \leqq n\}$. A problem is degenerate if and only if some feasible solution has a graph which is disconnected.

Theorem 5. For a general $m \times n$ transportation problem $(m \leqq n)$, the minimum number of vertices is

$$\frac{n!}{(n-m+1)!}$$

and this is achieved in certain examples. For $m \times n$ non-degenerate problems the minimum number of vertices is n^{m-1}. This gives a complete solution for the minimum number of vertices.

Only partial results are known about the maximum number of vertices of the general $m \times n$ problem.

Theorem 6. For $m \times n$ transportation problems, the number of vertices of $T(a,b)$ is achieved by non-degenerate problems.

Sketch of proof. Each $T(a,b)$ is the intersection of a positive orthant with a flat. This polytope can clearly be approximated as closely as desired by a polytope which represents a non-degenerate problem. One then makes use of the fact that the number of vertices of a polytope is a lower semi-continuous function of the polytope.

We next consider the maximum number of vertices for regular (= non-degenerate) problems.

Lemma 7. Two vertices of $T(a,b)$ which have the same pattern of non-zero entries must be identical.

Proof. Vertices x and y were characterized as matrices without loops. If x and y have the same non-zero pattern, then so must $\frac{1}{2}(x+y)$. Clearly this cannot happen unless $x = y$ because $\frac{1}{2}(x+y)$ must also be a vertex.

Since every extreme point of $T(a,b)$ has a graph which is a tree, we can get upper estimates of the number of vertices by counting certain classes of trees.

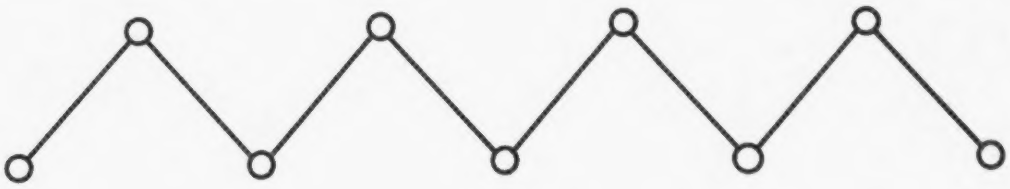
Theorem 8. For any $m \times n$ transportation problem the number of vertices is at most the number $m^{n-1}n^{m-1}$ of spanning trees of the complete bipartite graph with m vertices in one set, n in the other.

We can restrict ourselves to (connected) trees rather than forests (i.e., unions of disjoint trees) in the above theorem, because we have restricted the problem to the non-degenerate case. It should be noted that $m^{n-1}n^{m-1}$ is the number of "generalized" vertices of the d-polytope $T(a,b)$ in the following sense. Each genuine vertex is the intersection of d hyperplanes determined by d-facets. However the intersection of d hyperplanes may or may not be in the polytope and thus it might not be a vertex. The above number counts the number of such intersections and thus is only an upper bound on the number of genuine vertices. One could ask, how close to $m^{n-1}n^{m-1}$ vertices can you get with a transportation polytope? At present this seems to be a very hard problem, although some progress has been made.

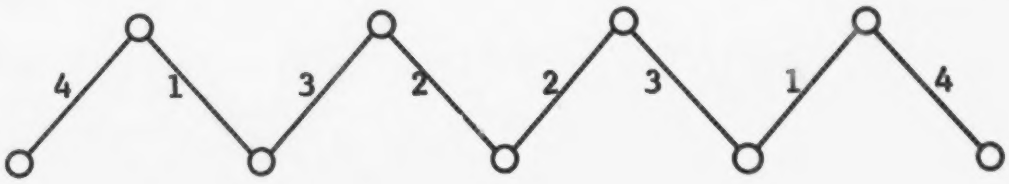
As an example, suppose we have n sources, each with supply $1/n$, and $n+1$ sinks, each with demand $1/(n+1)$. Then several observations may be made about the graph of a vertex of this polytope:

a) Each source must have at least two arcs leading to sinks, since $1/n > 1/(n+1)$.

b) Each source must have exactly two arcs leading to sinks, since any (connected) tree with $2n+1$ nodes must have exactly $2n$ edges. Thus the bipartite graph might look like this:



c) For every bipartite graph which is 2-valent at each source node, there is a feasible transportation scheme which may be assigned to the graph. For the graph shown above the assigned scheme will be



(where the amount k shown on each arc denotes that $k/n(n+1)$ of the total commodity moves along that arc).

d) The above arguments would work in a similar way if there were n sources, $\mu n + 1$ sinks, and therefore each source was $(\mu+1)$-valent.

e) The problem of determining the number of vertices to the transportation polytope when we impose these symmetry conditions is again a matter of counting trees. The number turns out to be $(n+1)^{n-1}(n!)$ when there are n equal source nodes and $n+1$ equal sink nodes, or

$$(\mu n+1)^{n-1} \quad \frac{(\mu n)!}{(\mu!)^n}$$

when there are n equal sources and $\mu n + 1$ equal sinks.

Our discussion of transportation polytopes will end with the following comment. Much of the material presented above is well-known but some of it is new. A much more careful exposition, accompanied by complete references and proofs, can be found in a forthcoming paper by V. Klee and C. Witzgall.

Discussion.

It was pointed out that the upper bound in Theorem 8 was frequently quite a bit too large. Several other problems were mentioned which appear similar to the transportation problem, but which apparently are related only in that combinatorial techniques are needed.

The symmetry of the last transportation problem example (n equally strong sources, $n+1$ equally strong sinks) leads to a question about the symmetry of the corresponding polytope $T(a,b)$ and the determination of its facial structure. Klee mentioned that Grünbaum would consider the problem of determining the number of facets of a d-polytope with v vertices, and remarked that it would be interesting to have similar results for the class of polytopes which had a group of symmetries of a particular order. Grünbaum remarked that almost no results of this type are known. Klee pointed out that theorems on the facial structure of polytopes are of direct interest to linear programming problems.

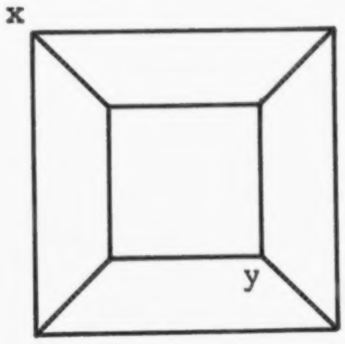
Woolf asked how proponents of a geometry course based on convexity, as Grünbaum and Klee have discussed, would meet the claim that geometry courses based on algebraic topology and differential geometry would be of more use to a student in his further work. Klee was of the opinion that to do something well in those two areas, a much better background and level of maturity is necessary than to do a careful treatment of convexity. In particular many topics in convexity are more intuitively accessible and elementary. If a student were going on to research in those areas, then a thorough background at the undergraduate level could be preferable, but for an undergraduate general training, a course based on convexity would, in Klee's opinion, be better. Paul Kelly pointed out that in planning this conference, it was felt that the place college geometry was least neglected was in differential geometry, and that it would be better to invite lecturers who would concentrate on the other parts of geometry which are in more trouble at the college level.

Lecture V.

We will next consider another topic related to linear programming. Perhaps this should be called a problem whose answer would lead to an application rather than an application of geometry. Much of this lecture can be found in the last chapters of Grünbaum's book, Convex Polytopes.

Linear programming problems are frequently solved by choosing a vertex of the polytope described by the linear constraints, investigating the neighboring vertices, and thus searching from one vertex to the next for the maximizing vertex. In such a search process we move along the edges from one vertex to the next, and it would be useful to have results which give upper bounds to the number of such iterations that a computer might have to perform. (The usefulness of a computer in solving problems often depends upon whether you know the problem will take at most 2 minutes, 2 days, or 2 years!)

Definition. Let G be a graph (for example, the set of edges and vertices of a polytope). The distance $\delta(x,y)$ between vertices x and y is the length of the shortest path (counting edges) between x and y. The diameter of G is $\max\{\delta(x,y) \mid x,y \text{ vertices of } G\}$. If P is a polytope, $\delta(P)$ denotes the diameter of the graph of P. For example, if P is a cube, then $\delta(P) = 3$.



Definition. $\Delta_b(d,n)$ is the maximum diameter of a d-polytope with n facets. (The b is for bounded; if the b is omitted it is the same definition for the class of (possibly unbounded) polyhedra.)

Definition. A graph G is d-connected if it cannot be separated between any two vertices by removing fewer than d vertices and their adjoint edges.

For example, the graph of the cube shown above is 3-connected. (See Grünbaum's lectures, Section 12, for further related results.)

Before stating the main theorems, we make a few observations.

Lemma 1. The graph of a d-polytope is d-connected. Equivalently, for each pair of distinct vertices of a d-polytope, there exist d disjoint paths connecting these vertices with only the end-points in common.

The proof is omitted.

Lemma 2. (Grünbaum-Motzkin) The maximum diameter of a d-polytope with v vertices is

$$[\frac{v-2}{d}] + 1.$$

Proof. Suppose x and y are vertices. Consider d independent paths from x and y and let k be the length of the shortest. Internal to each path there must be at least $k-1$ vertices. Thus $v \geqq (k-1)d + 2$, so $k-1 \leqq \frac{v-2}{d}$ and $k \leqq [(v-2)/d] + 1$.

Lemma 3. $\Delta_b(d,n)$ is achieved by a simple d-polytope with n facets. (P is simple if each vertex is d-valent.)

Proof sketch. A small amount of "wiggling" of each face of the polytope will not increase the number of faces, but can make each vertex d-valent, and in general will produce additional vertices and edges out of old multi-valent vertices. Thus P may be assumed to be simple.

This is useful in the 3-dimensional case.

Lemma 4. If $d = 3$, $\Delta_b(3,n) = [\frac{2n}{3}] - 1$.

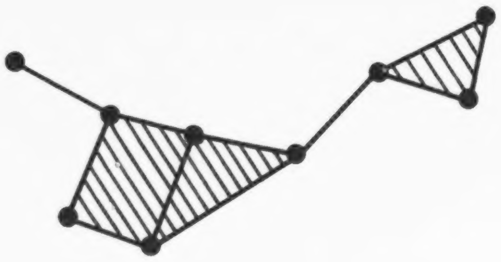
Proof. Letting v, e, f denote the number of vertices, edges, and faces, $v - e + f = 2$ by Euler's formula. P is simple implies that $2e = 3v$. Thus $v = 2f - 4$. By Lemma 2 and 3, $\Delta_b(3,f) \leqq [(v-2)/3] + 1 = [2f/3] - 1$. Easy examples show the bound is actually assumed.

Theorem. $\Delta_b(d,n) = [\frac{(d-1)n}{d}] - d + 2$ when $d \leqq 3$ or when $n \leqq d+5$, except that $\Delta_b(4,9) = 5$.

The proof is omitted. These give the only specific values of $\Delta_b(d,n)$ that are known.

The determination of $\Delta_b(d,2d)$ was a long outstanding problem in linear programming, and the conjecture that $\Delta(d,2d) = d$ was "proved" for $d \leqq 5$ before it was recently shown to be false even in E^4. Thus problems of this type are difficult even though they are very easily stated.

The conjecture of Hirsch was that $\Delta(d,n) \leqq n - d$. This has been shown false even in E^4. However the bounded version of the conjecture $\Delta_b(d,n) \leqq n-d$ is still open. A related conjecture that is very easy to state, and yet unsolved is: Any two vertices of a polytope can be joined by a path which does not "revisit" any facet (i.e., considering a path as a sequence of vertices, a path may travel around the vertices of a facet as far as desired, but once it leaves a facet it may never revisit the facet later in the sequence). If the latter conjecture were true, then the bounded version of Hirsch's conjecture would be true. Actually, the following stronger form of the conjecture might be true. In any cell-complex any two vertices may be connected by a path not revisiting any cell. It is not known whether this conjecture is true even for 2-dimensional cell complexes embedded in E^3. I would expect that these conjectures are probably false when d gets large, say $d \geqq 9$.



Details of these and similar results and problems may be found in Grünbaum's book.

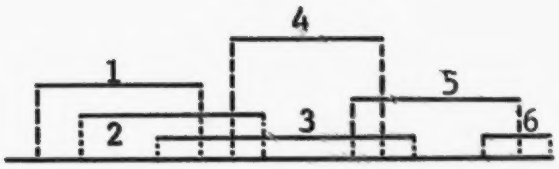
Applications of Helly's Theorem.

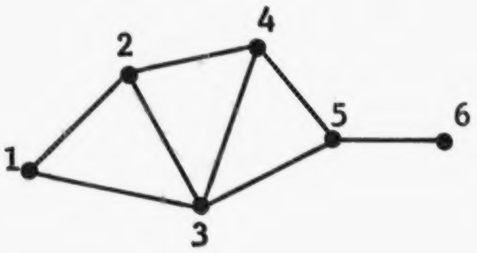
There are several applications to science of Helly's theorem or related results (see Grünbaum's lectures, Section 5) which could well be brought into a classroom discussion on Helly's theorem. One of these has to do with molecular genetics.

In studying DNA molecules which specify the structure of certain phage particles, the following investigation procedure was used. The phage particles exist both in certain standard forms and in mutant forms. Each mutant arises from some blemish in the genetic structure. If the structure of the DNA molecule is represented by a graph, we are interested in knowing the structure of this graph, and particularly in the part of the graph which contains the blemished part which produces the mutant forms. There seemed to be experimental reasons for assuming that the blemished portion of each graph was a sub-arc of the graph rather than just some random subset. Further there was a way of telling when two of the blemished sub-arcs overlapped. The researchers then asked whether this observed intersection pattern is consistent with the hypothesis that all blemished intervals came from a linear part of the graph of the DNA molecule. Or is it necessary to use more complicated structures than a line to observe the intersection pattern?

This leads to the following mathematical problem. Determine completely what intersection patterns can arise from finite families of intervals on a line. Helly's theorem on the line gives one limitation at once; namely, if a finite family of intervals on the line is such that each two have a common point then their intersection is non-empty. But this does not apply directly here for we are concerned only with pairwise intersections. The intersection pattern itself could be thought of as a square matrix of zeros and ones, with 0 in the (i,j)-place when $I_i \cap I_j = \emptyset$ for the mutants i and j, and 1 in the (i,j)-place if $I_i \cap I_j \neq \emptyset$. The question then becomes, is this matrix consistent with the assumption that the intervals are chosen from a line?

A more convenient way of describing this is in graph-theoretic language. The intersection graph of any finite family of sets is the graph whose nodes represent the sets, and two nodes are connected by an edge if and only if the corresponding sets have non-empty intersection. For example, the intersection graph of the line segments on the line at the left is the graph at the right.





We will say that a graph is an interval graph if it is the intersection graph of a finite number of intervals on the line. The problem is now to characterize all interval graphs in a useful way. We will state only one of the more satisfying characterizations of interval graphs. Most parts of the proof of this theorem would be understandable and interesting to undergraduates. A graph has the rigid circuit property if each circuit is decomposable into triangles, that is, each circuit with more than 3 edges has crossings.

Theorem. G is an interval graph if and only if

(1) it has the rigid circuit property, and

(2) each three vertices of the graph admit an ordering so that every path from vertex 1 to vertex 3 either goes through vertex 2 or is only one edge away from vertex 2.

To finish the story, it was shown that for the experimental results which were obtained, the intersection graph of the several hundred blemished intervals in the DNA molecules did indeed form an interval graph. Examples of this sort point out to students that there are useful and interesting things that can be said even about the intersection properties of convex sets on a line.

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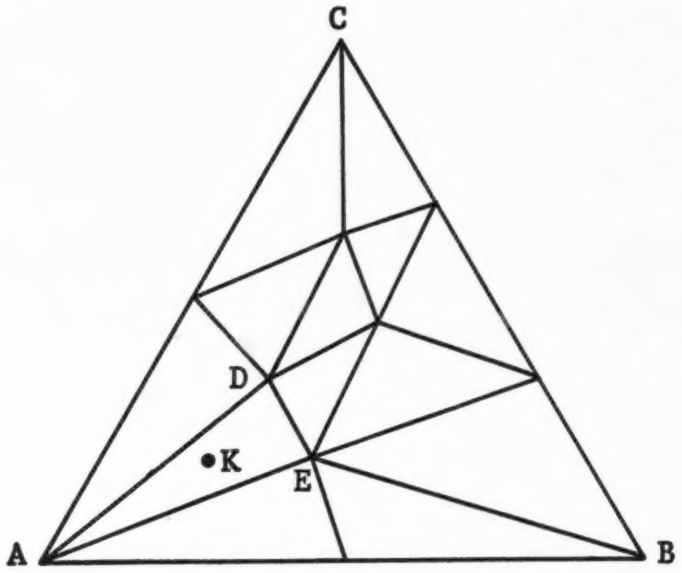
An application of barycentric coordinates.

This application of geometry is of interest in connection with mineral engineering. It deals with the phase diagram of a multicomponent situation which is isobaric, isothermal, and in which no solid solutions are formed. For the sake of illustration, let us consider an example with just three primary components, labeled A, B, C. Suppose we are interested in all the possible chemical compounds which could be formed from combining different amounts of these three primary components and which would be stable at the given pressure and temperature. Each such compound D can be represented as a point in the triangle ABC, where the unique barycentric coordinates of D with respect to A, B, and C determine the relative amounts of components A, B, and C that are present in D. (Generally it is possible only in parts of inorganic chemistry to determine a compound in terms of the amounts of the elements of primary components from which it is formed. This cannot be done with carbohydrates, for example, but there are large areas within inorganic chemistry where this rule does apply.) Along with all the vertices in this triangle which represent the different possible stable compounds, one has a decomposition of the triangle into smaller triangles, as shown. Of course, for a given set of vertices there are many such decompositions possible, but in fact there is only one such decomposition which represents the following important aspect of the situation. Suppose we started with a particular mixture of the compounds A, B, and C, and the relative amounts of each determine the barycentric coordinates of the point K in the obvious way. If this mixture is heated to a high temperature, so that all sorts of decomposition take place, and then is slowly cooled to the original temperature and pressure, the resulting mixture does not consist of the original 3 compounds A, B, C. Rather almost all of the resulting mixture will consist of the compounds which are represented by the vertices of the small triangle AED which



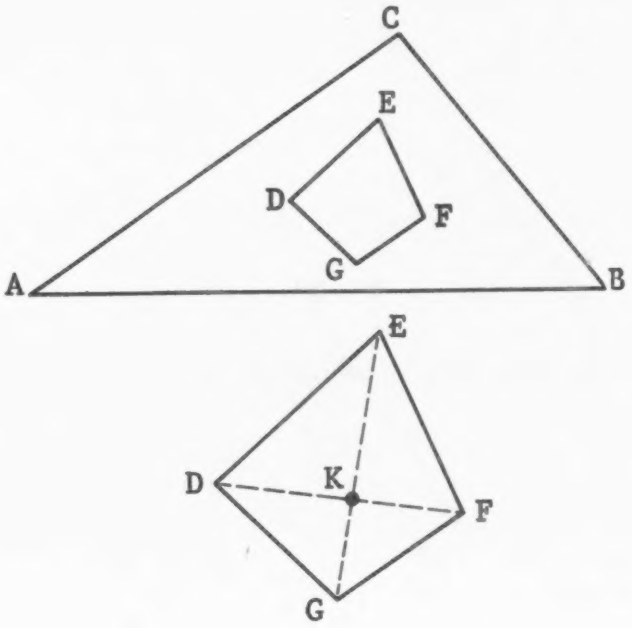
contains K. Furthermore, the relative amounts of the compounds A, E, and D will be determined by the barycentric coordinates of K in terms of A, E, and D.

It is easy to see the importance of this in certain refinery processes. Given a particular type of ore, for example, this type of scheme is useful in determining what kinds of compounds can be produced and in what amounts. This amounts to knowing the compounds in the ore (A, B, and C) and determining which small simplex (triangle) the ore (point K) lies in, and what its barycentric coordinates are. This is rather trivial in the 2-dimensional case since you can just draw a picture and see what is going on. In fact, this is just what people do. In the higher dimensional cases, barycentric coordinates must be computed in the standard ways. The problem mathematically, is the following. Let $p_0,p_1,\ldots,p_d$ be the vertices of a d-dimensional simplex which contains the smaller simplex whose vertices are $q_0,q_1,\ldots q_d$, and suppose $z \in \text{conv}(p_0,p_1,\ldots,p_d)$. We wish to know whether $z \in \text{conv}(q_0,q_1,\ldots q_d)$, and if so, what are the affine coordinates of z in terms of the q_i.

This sort of problem shows the student that there really are situations for which the actual computation of barycentric coordinates is desirable and a natural thing to do. The first ideas in computing barycentric coordinates are just the usual techniques with quotients of determinants, but there are practical refinements of these techniques which soon lead to the attempt to compute the coordinates in the most efficient manner.

Discussion.

It was asked how the decomposition of the triangle into smaller triangles was determined from knowing just the 3 original vertices and the locations of the interior vertices. Suppose for example, that you knew from the experiment that DEFG was a part of the diagram, and you wondered which of the two diagonals DF or GE should be added. The standard technique is to form a mixture whose relative amounts determine the coordinates of the point K, heat it, then cool it, and observe whether compounds E and G or compounds D and F result. Analogous techniques exist for the higher dimensional cases.



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Johnson pointed out that another application of barycentric coordinates has been in the partitioning of an integer number of legislature seats among various political parties, each of which has received a certain percentage of the total vote. In this setting the political parties determine the vertices of a simplex, and the simplex is subdivided into areas which represent certain divisions of the legislature. An article describing this is in Mathematical Snapshots by Steinhaus.

CONVEX SETS AND THE COMBINATORIAL THEORY OF CONVEX POLYTOPES

Lectures by Branko Grünbaum

(Lecture notes by John Reay)

Convexity is a subject that can easily be taught to undergraduates at the junior or senior levels, and there are several good reasons for doing so. First, it is possible to reach significant results without the necessity of a strong background in other fields. There is no prerequisite of any extensive techniques. Secondly, convexity is a tool which may be applied in many other fields of mathematics, including number theory, functional analysis, complex variables, and others. Also it develops a student's geometric intuition and intuitive comprehension of the proofs and theorems. It introduces combinatorial reasoning which has application in many other fields. One of its main advantages, which is rare in other areas of geometry, is that it is possible to introduce students to many open problems early in the course. This makes it possible to show that there is much work yet to be done, and problems that have challenged our best efforts to find a solution. This type of course can be quite a contrast to many courses which would lead the student to believe that all mathematics is done and comes in a completed package. Finally, many proofs of convexity have arguments in 2- and 3-dimensional space which makes for easy understanding, even though the proofs are valid in higher dimensions.

These notes cover only a small fraction of the directions in which convexity has developed. In the first part of the notes (Sections 1-9) a basic introduction is given to convexity and some of its main tools, applications, and references. Sample proofs are included, but many of the results are left in the form of unproved statements or exercises. These sections are not meant to be a complete review of the topics usually covered in a course on convexity, but they are rather meant to supply a foundation for the combinatorial theory of convex polytopes, one of the rapidly expanding areas of convexity. This second part of the notes (Sections 10-14) is primarily designed to give an intuitive grasp of the important results, open problems and basic techniques of this area.

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1. Algebraic prerequisites.

The following facts, generally known from algebra and easily proved, will be used--without special mention--in the sequel.

If X is a vector space over the reals R, and if $A = (a_1,\ldots,a_n)$ is a sequence of n vectors in X, any expression of the type $\Sigma_{i=1}^{n} \lambda_i a_i$, where each $\lambda_i \in R$, is a linear combination of the vectors in A. We shall say that A is linearly independent provided $\Sigma_{i=1}^{n} \lambda_i a_i = 0$ implies $\lambda_1 = \cdots = \lambda_n = 0$; if $\Sigma_{i=1}^{n} \lambda_i a_i = 0$ is possible with some $\lambda_i \neq 0$, we shall say that A is linearly dependent. Hence A is linearly dependent if and only if some member of A is a linear combination of the other members of A.

A maximal linearly independent sequence is called a linear basis of X; this clearly means that each vector $x \in X$ is a linear combination of the elements of a linear basis of X. Moreover, this linear combination is unique; its coefficients are the coordinates of x relative to the given linear basis.

Any two bases of the same space X have the same cardinality, known as the (linear) dimension of X. If X is d-dimensional for some finite cardinal d, and if $A = (a_1,\ldots,a_d)$ and $B = (b_1,\ldots,b_d)$ are two linear bases of X, there exists a regular linear transformation T of X onto itself such that $Ta_i = b_i$ for $i = 1,\ldots,d$. Conversely, if A is a linear basis of X and T is a regular linear transformation of X onto itself, then $(Ta_1,\ldots,Ta_d)$ is a basis for X. Any linearly independent sequence in X may be extended to a linear basis for X.

If X is d-dimensional, a sequence $A = (a_1,\ldots,a_n)$ of vectors in X is linearly independent if and only if the matrix

$$\begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1d} \\ \cdot & \cdots & \cdot \\ \alpha_{n1} & \cdots & \alpha_{nd} \end{pmatrix}$$

has rank n, where $\alpha_{i1},\ldots,\alpha_{id}$ are the coordinates of a_i relative to a given linear basis of X.

The linear dimension of the Euclidean d-space E^d is d. A convenient linear basis of E^d is the standard basis formed by the vectors $e_1 = (1,0,\ldots,0)$, $e_2 = (0,1,0,\ldots,0),\ldots,$ $e_d = (0,\ldots,0,1)$.

If X and X* are two vector spaces of dimension d, and if $x_1,\ldots,x_d$ form a basis of X, and $x_1^*,\ldots,x_d^*$ form a basis of X*, a mapping T from X to X* may be defined by $T(\Sigma_{i=1}^d \lambda_i x_i) = \Sigma_{i=1}^d \lambda_i x_i^*$. This mapping T is an algebraic isomorphism between X and X*; moreover, if both X and X* are topological vector spaces, T is a homeomorphism.

The scalar product $\langle a,b\rangle$ of vectors $a,b \in E^d$ is the real number defined by

$$\langle a,b\rangle = \Sigma_{i=1}^d \alpha_i\beta_i \ .$$

The most important properties of the scalar product are

$$\langle a,b\rangle = \langle b,a\rangle$$

$$\langle \lambda a,b\rangle = \lambda\langle a,b\rangle$$

$$\langle a+b,c\rangle = \langle a,c\rangle + \langle b,c\rangle$$

$$\langle a,a\rangle \geqq 0 \text{ with equality if and only if } a = 0.$$

If $\langle a,b\rangle = 0$, then a and b are said to be orthogonal to each other. If $\langle a,a\rangle = 1$, then a is called a unit vector. In the sequel, the letter u (with or without subscripts) shall be used only for unit vectors.

A hyperplane H in E^d is a set which may be defined as $H = \{x \in E^d \mid \langle x,y \rangle = \alpha\}$, for suitable $y \in E^d$, $y \neq 0$, and α. An open halfspace (closed halfspace) is defined as $\{x \in E^d \mid \langle x,y \rangle > \alpha\}$ (respectively $\{x \in E^d \mid \langle x,y \rangle \geqq \alpha\}$) for suitable $y \in E^d$, $y \neq 0$, and α. Clearly, $\{x \in E^d \mid \langle x,y \rangle < \alpha\}$ is also an open halfspace for $y \neq 0$; similarly for closed halfspaces.

If X is a real vector space of finite dimension d, a norm $\| \ \|$ may be defined on X which turns X into a metric space isometric with the Euclidean d-space E^d. Thus each d-dimensional X is "essentially" the space E^d. (The norm in X is defined by $\|x\| = \sqrt{\langle x,x \rangle}$. In order to find an isometry between X and E^d, take any basis $x_1,\ldots,x_d$ of X and "orthomormalize" it by putting $y_1 = x_1$ and

$$y_k = x_k - \sum_{i=1}^{k-1} \frac{\langle x_k, y_i \rangle}{\langle y_i, y_i \rangle} y_i \quad \text{for } k = 2,\ldots,d.$$

Let $e_i' = y_i / \sqrt{\langle y_i, y_i \rangle}$ for $i = 1,\ldots,d$. Then $e_1',\ldots,e_d'$ form a linear basis of X and, if $e_1,\ldots,e_d$ is the standard basis of E^d, the transformation T such that $Te_i = e_i'$, $T(\Sigma_{i=1}^{d} \lambda_i e_i) = \Sigma_{i=1}^{d} \lambda_i e_i'$ is an isometry of the required type.)

The affine facts listed below may either be derived from their linear counterparts, or be proved independently.

If X is a vector space over the reals, and if $A = (a_1,\ldots,a_n)$ is a sequence of elements of X, any expression of the type $\Sigma_{i=1}^{n} \lambda_i a_i$, where all $\lambda \in R$ and $\Sigma_{i=1}^{n} \lambda_i = 1$, is an affine combination of the elements of A. We shall say that A is affinely independent provided no member of A is an affine combination of the other members of A; otherwise A is affinely dependent.

A is affinely dependent if and only if there exist $\lambda_1,\dots,\lambda_n$ in R, not all equal to 0, such that $\Sigma_{i=1}^n \lambda_i a_i = 0$ and $\Sigma_{i=1}^n \lambda_i = 0$.

$A = (a_1,\dots,a_n)$ is affinely dependent if and only if the sequence $(a_1 - a_n,\dots,a_{n-1} - a_n)$ is linearly dependent.

If X is of (linear) dimension d, and if the elements a_i of A have coordinates $\alpha_{i1},\dots,\alpha_{id}$ relative to some linear basis of X, then A is affinely independent if and only if the matrix

$$\begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1d} & 1 \\ \vdots & & \vdots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nd} & 1 \end{pmatrix}$$

has rank n.

Any maximal, affinely independent sequence in X is an affine basis of X. If A is a linear basis of X then $A \cup \{0\}$ is an affine basis of X.

The set of all affine combinations of two different points $x,y \in E^d$ is the line $L(x,y) = \{(1-\lambda)x + \lambda y \mid \lambda \text{ real}\}$. If $x',y' \in L(x,y)$ and $x' \neq y'$, then $L(x',y') = L(x,y)$.

If a set H has the property that $L(x,y) \subset H$ whenever $x,y \in H$, $x \neq y$, we call H a flat, or an affine variety. Clearly, the set of all affine combinations formed from all finite subsets of a given set A is a flat; it is denoted by $\operatorname{aff} A$ and is called the affine hull of A. The family of all flats in E^d contains E^d, $\emptyset$, all one-pointed sets, all lines, all hyperplanes; also, it is intersectional: if all H_α's are flats, so is $\cap_\alpha H_\alpha$. The affine hull $\operatorname{aff} A$ of a set A may equivalently be defined as the intersection of all flats which contain A. The formation of the affine hull is translation invariant; i.e., $\text{aff}(x+A) = x + \text{aff } A$. (Note that $\text{aff}(A+B) = \text{aff } A + \text{aff } B$ is not true in general.)

Every flat H is a translate $H = x + V$ of some subspace V of E^d, and is therefore isomorphic to the Euclidean space of a certain dimension $r \leq d$; the dimension of H (and of V) is then $r = \dim H = \dim V$.

Each r-dimensional flat contains $r+1$ affinely independent points, but each $(r+2)$-membered set of its points is affinely dependent.

A transformation T of E^d into E^n is affine provided it preserves affine combinations; that is,

$$T(\Sigma_{i=1}^{k} \alpha_i x_i) = \Sigma_{i=1}^{k} \alpha_i T x_i$$

whenever $x_i \in E^d$ and α_i's are reals satisfying $\Sigma_{i=1}^{k} \alpha_i = 1$.

Images, or inverse images, of flats under affine transformations are again flats. In particular, an affine transformation A from E^d to $E^1 = R$ is determined by $Ax = \langle a,x\rangle$. If $a \neq 0$, the inverse image by A of any point $\alpha \in R$ is a hyperplane in E^d, denoted by

$$H(a,\alpha) = \{x \in E^d \mid Ax = \langle a,x\rangle = \alpha\}.$$

Conversely, each $(d-1)$-flat in E^d is expressible in this form for suitable a and α and is, consequently, a hyperplane.

The main advantage of the affine notions over the corresponding linear ones is their invariance under translations. Hence, among other advantages, their use in proofs avoids "choosing the origin" at some advantageous point.

The assumption that X is a vector space over the reals is irrelevant to many of the definitions and facts listed above. Any field of characteristic

0 could be used instead of the real numbers.

Discussion.

It was generally felt that the affine notions (affine independence, transformations, etc.) were usually unfamiliar to the students. It was suggested that this could be due to a lack of affine topics in most linear algebra courses. Opinions about how much time was necessary to give an adequate foundation in affine topics varied from one or two hours upwards. Klee suggested letting the word "blank" stand for any of the words linear, affine, positive, or convex, and then develop the general ideas of "blank independence," "blank hulls," etc. as far as possible with comparisons of their differences. See Klee [1].

2. Definition and elementary properties of convex sets.

Let X be any vector (linear) space over the reals, of finite or infinite dimension. A set $K \subset X$ is called convex provided

(*) whenever $x,y \in K$, then all points of the (straight-line) segment determined by x and y belong to K.

Denoting by R the field of real numbers, condition (*) may be reformulated as

(**) whenever $x,y \in K$ and $\lambda \in R$ satisfies $0 \leqq \lambda \leqq 1$, then $\lambda x + (1-\lambda)y \in K$.

Examples of convex sets in any real vector space X:

(i) the empty set $\emptyset$; any single point; the whole space X:

(ii) any (linear) subspace of X and, more generally, any flat (i.e., translate of a subspace) in X.

(iii) any (closed or open) halfspace of X (i.e., any set of the type $\{x \in X \mid \varphi(x) \geqq \alpha\}$ or $\{x \in X \mid \varphi(x) > \alpha\}$, where $\alpha \in R$ and φ is a non-trivial distributive functional on X; this means that φ is real-valued, $\varphi(x) \neq 0$ for some $x \in X$, and $\varphi(\alpha x + \beta y) = \alpha\varphi(x) + \beta\varphi(y)$ whenever $x,y \in X$, $\alpha,\beta \in R$).

(iv) if X is a normed space, any closed or open ball in X (i.e., set of the type $\{x \in X \mid \|x-x_o\| \leqq \alpha\}$ or $\{x \in X \mid \|x-x_o\| < \alpha\}$, where $x_o \in X$, $\alpha \in R$, and $\alpha > 0$).

Convex sets are general enough to appear in widely different fields, from elementary geometry to functional analysis. On the other hand, they have sufficiently much structure to allow significant results.

In the sequel we shall mostly assume that X is the d-dimensional Euclidean space E^d, though many of the results do not require this restriction, or may be modified to hold in more general spaces.

We list now a few fundamental properties of convex sets in E^d; additional properties will be discussed later.

1. The family of all convex sets in E^d is intersectional; that means that for any non-empty family $\{K_\nu\}$ of convex sets in E^d, the intersection $\cap_\nu K_\nu$ is of the same type. All closed convex subsets of E^d also form an intersectional family, and so do all bounded convex sets, or all compact convex sets.

2. Any convex combination of points belonging to a convex set $K \subset E^d$ belongs to K. This means

(***) whenever $n \geqq 1$, $\alpha_1,\ldots,\alpha_n \in R$, $\alpha_i \geqq 0$ for $i = 1,\ldots,n$ and $\Sigma_{i=1}^n \alpha_i = 1$, then $x_1,\ldots,x_n \in K$ implies $\Sigma_{i=1}^n \alpha_i x_i \in K$.

Clearly, condition (**) used in defining convexity is the special case $n = 2$ of (***). On the other hand, (**) implies (***), as is easily seen by induction on n: assume without loss of generality that $\alpha_n \neq 1$, and use the identity

$$\sum_{i=1}^n \alpha_i x_i = \alpha_n x_n + (1-\alpha_n) \sum_{i=1}^{n-1} \frac{\alpha_i}{M} x_i,$$

where $M = \Sigma_{j=1}^{n-1} \alpha_j$.

3. If A is an affine transformation of E^d into $E^{d'}$, and if $K \subset E^d$ and $K' \subset E^{d'}$ are convex sets, then $AK = \{Ax \mid x \in K\}$ is a convex set, and so is the inverse image by A of K', $A^{-1}K' = \{x \in E^d \mid Ax \in K'\}$.

Hence, in particular, the convexity of a set K is not affected by any translation $(x+K)$, homothecy (αK), or orthogonal transformation. This may be used in proofs to reduce the more general situation to one in which a suitable point is at the origin. (See, for example, the proof of Theorem 6.10.)

4. If K_1 and K_2 are convex sets in E^d, then $K_1 + K_2 = \{x_1 + x_2 \mid x_1 \in K_1, x_2 \in K_2\}$ is a convex set.

Note that we shall use the notation $-K = (-1)K = \{-x \mid x \in K\}$ and $K_1 - K_2 = K_1 + (-K_2)$. Hence "subtraction" of sets $(K_1 - K_2)$ is not the inverse operation of the (vector, or Minkowski) "addition" of sets $(K_1 + K_2)$.

5. If $K \subset E^d$ is convex, then its topological closure $\mathrm{cl}\, K$ and its interior $\mathrm{int}\, K$ are also convex. Thus if $x,y \in \mathrm{int}\, K$ and $0 \leqq \lambda \leqq 1$, then $(\lambda x + (1-\lambda)y) \in \mathrm{int}\, K$.

Denoting the distance between points $x,y \in E^d$ by $\|x-y\| = \sqrt{\langle x-y,x-y\rangle}$, the convexity of $\mathrm{cl}\, K$ follows from the observation that if $\|x_i-x\| < \delta$ and $\|y_i-y\| < \delta$ then, for $0 \leqq \lambda \leqq 1$, $\|(\lambda x_i + (1-\lambda)y_i) - (\lambda x + (1-\lambda)y)\| \leqq \lambda\|x_i-x\| + (1-\lambda)\|y_i-y\| < \delta$. An alternate proof of the convexity of $\mathrm{cl}\, K$ follows from 1, 3, and 4 above and the observation that $\mathrm{cl}\, K = \bigcap_{\epsilon>0}(K + \epsilon B)$, where B is the unit ball centered at the origin. The convexity of $\mathrm{int}\, K$ follows from the fact that there is some open d-ball B centered at the origin for which $x + B \subset K$ and $y + B \subset K$. Thus if $0 \leqq \lambda \leqq 1$ and if $u \in B$ then

$$(\lambda x + (1-\lambda)y) + u = \lambda(x+u) + (1-\lambda)(y+u) \in K$$

and therefore $(\lambda x + (1-\lambda)y) + B \subset K$, so $\mathrm{int}\, K$ is convex.

6. Similarly, if $x \in \mathrm{int}\, K$ and $y \in K$ where $K \subset E^d$ is convex, and if $0 < \lambda \leqq 1$, then $\lambda x + (1-\lambda)y \in \mathrm{int}\, K$.

As in the last proof, if $x + B \subset K$ then it follows that $(\lambda x + (1-\lambda)y) + \lambda B \subset K$.

Remarks

1. The intersectionality of the family of all convex sets in E^d may be used to define convex sets, or certain types of convex sets. For example, we shall see in Section 3 that the family of all closed convex proper subsets of E^d may be characterized as the smallest intersectional family of subsets of E^d which contains all closed halfspaces. Similarly, the family of all compact convex subsets of E^d is the smallest intersectional family of subsets of E^d containing all closed d-balls. All convex proper subsets of E^d may be obtained analogously by starting with "semi-spaces". (Semi-spaces may be explicitly defined in various ways: a particularly appealing definition--though unsuitable if one intends to use semi-spaces to define convex sets--is: a semi-space is any maximal, convex set which does not contain a given point.)

Other instances of this approach to convexity will be discussed later.

2. Modifications of conditions (**) or (***) are frequently encountered in the definitions of various families of sets in Euclidean spaces, or in more general vector spaces. A few examples are:

√

	Type of set	Together with x and y the set also contains	Whenever $\alpha,\beta \in R$ and
(i)	(linear) subspace	$\alpha x + \beta y$	no other condition
(ii)	flat (affine variety)	$\alpha x + \beta y$	$\alpha + \beta = 1$
(iii)	cone	αx	$\alpha > 0$
(iv)	convex cone	$\alpha x + \beta y$	$\alpha \geqq 0,\ \beta > 0$
(v)	convex set	$\alpha x + \beta y$	$\alpha + \beta = 1,\ \alpha,\beta \geqq 0$

Linear, *affine* and *positive* combinations (generalizing (i), (ii), and (iv)) are defined similarly to the fashion in which convex combinations were defined by (***). See Klee [1]. Some of the results on convex combinations we shall see later have analogues for the other types of combinations (or for some of them).

A number of other notions related to the above have been studied; such as, for example, the *mid-point convexity* (where with x and y the set contains $\frac{1}{2}(x+y)$). Some results are known on combinations with arbitrarily prescribed or restricted sets of coefficients. See the article by L. Danzer, B. Grünbaum, and V. Klee [1] -- referred to hereafter as DGK [1] -- and Motzkin [1].

3. It is sometimes convenient to consider *projective transformations* of convex sets in E^d. A transformation P from E^d to itself is called *projective* provided it is possible to express P in the form

$$Px = \frac{Ax + b}{\langle c,x\rangle + \delta},$$

where A is a linear transformation of E^d into itself, $b,c \in E^d$, $\delta \in R$, and at least one of c and δ is different from 0.

The transformation P is not defined on the set

$$N(P) = \{y \in E^d \mid \langle c,y\rangle + \delta = 0\}.$$

The set $N(P)$ may be empty (if $c = 0 \neq \delta$), in which case P is an affine transformation. If $c \neq 0$ then $N(P)$ is a hyperplane (which, if A is regular, is mapped by P onto the "hyperplane at infinity" of the projective d-space containing E^d).

If K is a set in E^d and P a projective transformation as above, P is said to be permissible for K provided $K \cap N(P) = \emptyset$. Generalizing property 3 we have:

If $K \subset E^d$ is a convex set and if P is a projective transformation permissible for K, then PK is convex.

4. The notion of convexity has been modified in numerous ways to make it suitable for settings different from vector spaces over the reals. In particular see DGK [1], Section 9 especially. We shall mention here a few of these variants; it is rather instructive to investigate the results we shall encounter with respect to the modifications needed in their proofs to make them valid for the different "convexities."

(i) If Q is any ordered field, convexity in any vector space over Q may be defined by the analogue of (**). This notion is useful in clarifying the role of completeness, closure, or compactness assumptions in various theorems. A particularly interesting special case is that in which finite-dimensional spaces over the field of rational numbers are considered.

(ii) Experience has shown that the most useful definition of convexity in the d-dimensional projective space P^d is:

A subset $K \subset P^d$ is convex provided there exists a (d-1)-dimensional subspace of H of P^d such that $K \cap H = \emptyset$, and if for each straight line L of P^d the set $K \cap L$ is either empty or connected.

Note that this setting is not interesting if we are dealing with a single convex set in P^d since a projective transformation carries the hyperplane H onto the hyperplane at infinity and the set K onto a set equivalent to a set in E^d. Note that property 1 does not hold for projective convexity.

(iii) A number of notions of spherical convexity have been found useful. (See p. 157 of DGK [1] for four different definitions and further references.) The one we shall have occasion to mention later is the following. Let S^d denote the unit d-sphere in E^{d+1}, that is, $S^d = \{x \in E^{d+1} \mid \langle x,x \rangle = 1\}$. A set $K \subset S^d$ will be called convex provided the set $\text{cone}_0 K \subset E^{d+1}$ is convex, where $\text{cone}_0 K$ denotes the union of all closed rays (half-lines) originating at the origin $0 \in E^{d+1}$ and passing through a point of K.

Exercises

1. Prove in detail the assertions made above.
2. Prove the convexity of the (closed) unit d-ball $B^d = \{x \in E^d \mid \|x\| \leqq 1\} \subset E^d$.
3. Prove the convexity of every d-simplex.
4. Show that each closed, midpoint convex subset of E^d is convex.
5. Let $\{K_\nu\}$ be a family of convex sets in E^d. Show that if every denumerable subfamily of $\{K_\nu\}$ has a non-empty intersection, then $\cap_\nu K_\nu \neq \emptyset$.
6. Let $K', K'' \subset E^d$ be called "equivalent" provided there exist $x \in E^d$ and $\lambda > 0$ such that $K'' = x + \lambda K'$. Prove that this is an equivalence relation, and that there are eleven distinct equivalence classes of convex subsets of $E^1 = R$. What is the cardinal of the distinct classes of convex subsets of E^2?

7. Let $K \subset E^d$ have the origin 0 as center of symmetry (i.e., $K = -K$), and assume that K is compact and that $\operatorname{int} K \neq \emptyset$. Show that the function φ defined for $x \in E^d$ by

$$\varphi(x) = \inf\{\alpha \geqq 0 \mid x \in \alpha K\}$$

is a norm on E^d. (That is

(i) $\varphi(x) \geqq 0$ for all $x \in E^d$, with $\varphi(x) = 0$ if and only if $x = 0$;

(ii) $\varphi(x+y) \leqq \varphi(x) + \varphi(y)$ for all $x,y \in E^d$

(iii) $\varphi(\alpha x) = |\alpha| \varphi(x)$ for all $x \in E^d$ and all $\alpha \in R$.)

Equivalently, show that $\rho(x,y) = \varphi(x-y)$ is a metric on E^d.

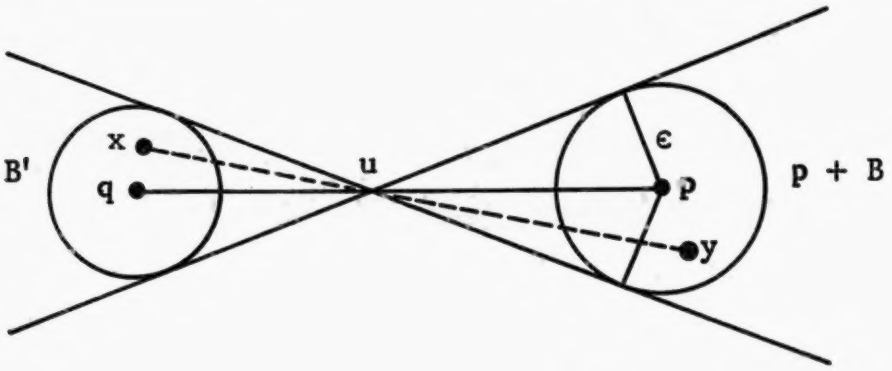
8. If K is a compact convex subset of E^d and B is the (topological) boundary of K, and if $C = \{\frac{1}{2}x + \frac{1}{2}y \mid x,y \in B\}$, then $K = C$. The example of a closed halfspace shows that "compact" may not be weakened to "closed."

Discussion.

Johnson pointed out that one of the various definitions of spherical convexity is equivalent to convexity in P^d, and raised the question of defining convexity in vector spaces over finite fields. This appears to be a difficult problem.

Klee suggested the following proof of a slightly stronger version of 6: If $q \in \operatorname{cl} K$, $p \in \operatorname{int} K$, and K is convex, then $u = \lambda q + (1-\lambda)p \in \operatorname{int} K$, for $0 \leqq \lambda < 1$.

As before, assume B is an open ball at the origin of radius ϵ for which $p + B \subset K$. Consider the open ball B' centered at q of radius



$\frac{\lambda}{1-\lambda} \cdot \epsilon$, that is, the ball which is the reflection through the point u. Then $q \in \mathrm{cl}\, K$ implies that there is some $x \in K \cap B'$. Then u lies in the relative interior of the line segment determined by x and some $y \in p + B$. The proof then proceeds as in 6.

3. Separation and support.

Let K denote a convex subset of E^d, $d \geqq 0$. We shall first prove:

1. $\text{int } K \neq \emptyset$ if and only if K contains $d+1$ affinely independent points.

In other words, the interior of K is non-empty if and only if the affine hull $\text{aff } K$ of K is the whole space E^d.

Indeed, if a maximal affinely independent set of points of K contains d or fewer points, their affine hull contains K but has dimension $d-1$ or less--hence $\text{int } K = \emptyset$. On the other hand, if K contains affinely independent points $x_0,\ldots,x_d$, then K contains the open simplex

$$\{\Sigma_{i=0}^{d} \lambda_i x_i \mid \Sigma_{i=0}^{d} \lambda_i = 1, \ \lambda_0 > 0,\ldots,\lambda_d > 0\}$$

hence $\text{int } K \neq \emptyset$.

For $A \subset E^d$ let $\text{rel int } A$ and $\text{rel bd } A$ denote, respectively, the interior and the boundary of A considered as a subset of the flat $\text{aff } A$ (which is isometric to some E^k). From 1 there follows

2. $\text{rel int } K \neq \emptyset$ whenever $K \neq \emptyset$.

Indeed, the following stronger result holds:

3. $\text{int } K \subset \text{rel int } K = \text{rel int } (\text{cl } K) \subset K \subset \text{cl}(\text{rel int } K) = \text{cl } K$.

The proof of 3 uses 2 (or 1) and the technique used in the proof of 2.5 (i.e., result 5 in Section 2).

Let $A_1, A_2 \subset E^d$; we shall say that A_1 and A_2 are separated by a hyperplane $H(u,\lambda) = \{x \in E^d \mid \langle u,x\rangle = \lambda\}$ (where $u \in E^d$ is a unit vector, and $\lambda \in R$) provided A_1 is contained in one and A_2 in the other of the closed halfspaces $H^+ = H^+(u,\lambda) = \{x \in E^d \mid \langle u,x\rangle \geqq \lambda\}$ and $H^- = H^-(u,\lambda) = \{x \in E^d \mid \langle u,x\rangle \leqq \lambda\}$. Note that a set may be separated from itself, for example, a line segment in E^2. We shall say that non-empty sets A_1 and A_2

are strictly separated by $H(u,\lambda)$ provided A_1 is contained in one and A_2 in the other of the open halfspaces $\operatorname{int} H^+$ and $\operatorname{int} H^-$.

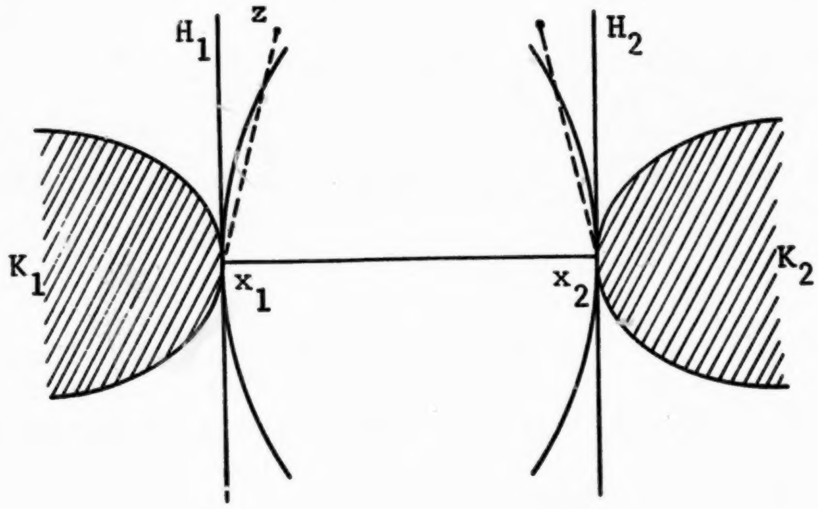
It is immediate that

4. If $H = H(u,\lambda)$ separates A_1 and A_2, then H separates $\operatorname{cl} A_1$ and $\operatorname{cl} A_2$, and it strictly separates $\operatorname{int} A_1$ and $\operatorname{int} A_2$.

The fundamental lemma on strict separation is

5. If K_1 and K_2 are compact convex sets in E^d such that $K_1 \cap K_2 = \emptyset$, there exists a hyperplane which strictly separates K_1 and K_2.

Proof. Let $\alpha = \inf \{\|y_2 - y_1\| \mid y_1 \in K_1,\ y_2 \in K_2\} = \delta(K_1,K_2)$. Since K_1 and K_2 are compact and disjoint, $\alpha > 0$ and there exist $x_1 \in K_1$ and $x_2 \in K_2$ such that $\|x_2 - x_1\| = \alpha$. Then, if H_1 denotes the hyperplane $H(x_1 - x_2, \langle x_1, x_1 - x_2 \rangle)$, and if $H_2 = H(x_1 - x_2, \langle x_2, x_1 - x_2 \rangle)$, the convexity of K_i implies that $K_1 \subset H_1^+ \subset \operatorname{int} H_2^+$ and $K_2 \subset H_2^- \subset \operatorname{int} H_1^-$.



Hence, for any λ such that $0 < \lambda < 1$, the hyperplane $H(x_1 - x_2, \langle \lambda x_1 + (1-\lambda)x_2, x_1 - x_2 \rangle)$ strictly separates K_1 and K_2.

Indeed, if some point z of K_1 were in the interior of H_1^-, then clearly some point on the line segment $[z,x_1]$ would be in K_1 by convexity, and be closer to x_2 than x_1. Note that this argument is 2-dimensional, even though the proof is valid in E^d. Also note that this is the only point where the convexity hypothesis enters. The only use made of compactness was to obtain the existence of x_1 and x_2.

Regarding the separation we have

6. If K_1 and K_2 are bounded convex sets in E^d such that $\text{aff}\,(K_1 \cup K_2) = E^d$, then K_1 and K_2 may be separated by a hyperplane if and only if

$$\text{rel int } K_1 \cap \text{rel int } K_2 = \emptyset.$$

The "only if" assertion is an easy exercise; we shall prove here only the "if" part of the theorem, assuming without loss of generality (see 3 and 4) that K_1 and K_2 are closed (hence compact). For $0 < \varepsilon < 1$, let $K_1(\varepsilon) = x_0 + (1-\varepsilon)(-x_0 + K_1)$ and $K_2(\varepsilon) = y_0 + (1-\varepsilon)(-y_0 + K_2)$, where $x_0 \in \text{rel int } K_1$ and $y_0 \in \text{rel int } K_2$ are fixed. Then

$$\text{rel int } K_i = \bigcup_{0 < \varepsilon < 1} K_i(\varepsilon) \quad \text{for} \quad i = 1,2,$$

$$K_i(\varepsilon') \subset \text{rel int } K_i(\varepsilon) \quad \text{for} \quad 0 < \varepsilon < \varepsilon' < 1 \quad \text{and} \quad i = 1,2.$$

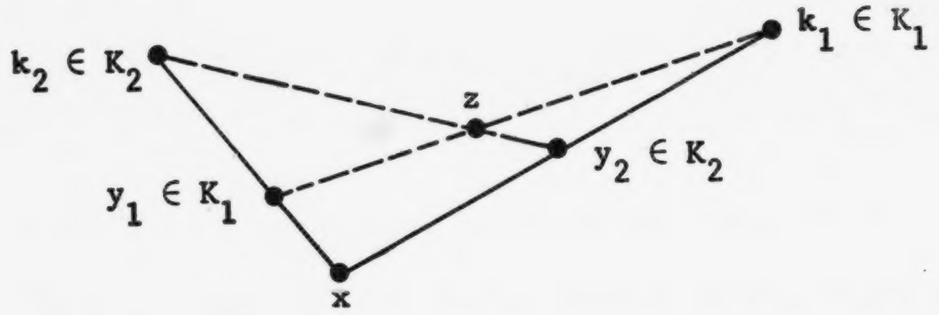
Since $K_1(\varepsilon)$ and $K_2(\varepsilon)$ are compact and disjoint, 5 implies the existence of a hyperplane $H(\varepsilon) = H(u(\varepsilon), \lambda(\varepsilon))$ which strictly separates $K_1(\varepsilon)$ and $K_2(\varepsilon)$. Since each $H(\varepsilon)$ intersects the segment $[x_0,y_0]$, the set $\{\lambda(\varepsilon) \mid 0 < \varepsilon < 1\}$ is bounded. Together with the compactness of the unit sphere S^{d-1} which contains the set $\{u(\varepsilon) \mid 0 < \varepsilon < 1\}$, this implies the existence of a sequence $(\varepsilon_1,\ldots,\varepsilon_n,\ldots)$, $\lim_n \varepsilon_n = 0$, for which the following limits exist $\lambda = \lim \lambda(\epsilon_n)$ and $u = \lim u(\epsilon_n)$. Then the hyperplane $H(u,\lambda)$ is easily seen to separate rel int K_1 and rel int K_2, hence also K_1 and K_2.

An alternate proof of 5 can be given based upon the following assertion: If C_1 is a non-empty convex proper subset of E^d and $C_2 = E^d \sim C_1$ is also convex, then cl C_i is a (closed) halfspace and bd C_1 = bd C_2 = a hyperplane. (See Hammer [1].) Assume K_1 and K_2 are disjoint convex sets in E^d and x is any point of E^d. We use the following Lemma: Either $K_1 \cap \operatorname{conv}(\{x\} \cup K_2) = \emptyset$, or else $K_2 \cap \operatorname{conv}(\{x\} \cup K_1) = \emptyset$, where conv A denotes the smallest convex set which contains A. (See Section 4.)

Well-order all points in the space, and with this Lemma add x_1 to one of the two sets K_i to obtain disjoint convex sets K_{11}, K_{21}, one of which contains x_1, and for which $K_i \subset K_{i1}$. In a similar fashion expand these two sets to two new disjoint convex sets K_{12}, K_{22}, one of which contains x_2. By transfinite induction, we obtain two disjoint convex sets K_1^* and K_2^* whose union is E^d and for which $K_i \subset K_i^*$. By the first assertion these sets are separated by the hyperplane which forms their common boundary. Thus it suffices to establish the lemma. Its denial asserts that for some point x, $K_1 \cap \operatorname{conv}(\{x\} \cup K_2) \neq \emptyset$ and $K_2 \cap \operatorname{conv}(\{x\} \cup K_1) \neq \emptyset$. Thus there must be points $k_i \in K_i$ and a point $x \in E^d$ and points

$$y_1 \in K_1 \cap \text{rel int conv } \{x,k_2\}$$

$$y_2 \in K_2 \cap \text{rel int conv } \{x,k_1\}.$$



This two-dimensional situation clearly leads to the existence of a point $z \in K_1 \cap K_2$, a contradiction. Thus 5 holds.

Both of 5 and 6 may be strengthened by weakening the hypotheses, but the basic ideas of these proofs already appear in the above formulations. In 5, for example, we may demand that only one of the closed sets K_i be compact. Another application of the compactness argument (relative to $u(\varepsilon)$ and $\lambda(\varepsilon)$) allows one to strike "bounded" from 6 and to establish

7. If K_1 and K_2 are convex sets in E^d such that $\operatorname{aff}(K_1 \cup K_2) = E^d$, then K_1 and K_2 may be separated by some hyperplane if and only if $\operatorname{rel\,int} K_1 \cap \operatorname{rel\,int} K_2 = \emptyset$.

As simple consequences of the above we have

8. Each closed convex set K in E^d is the intersection of all the closed (or of all the open) halfspaces of E^d which contain K. Each open convex set K in E^d is the intersection of all the open halfspaces of E^d which contain K.

9. If K is a convex set in E^d and if C is a convex set such that $C \subset \operatorname{bd} K$, then there exists a hyperplane which separates K and C.

* * * *

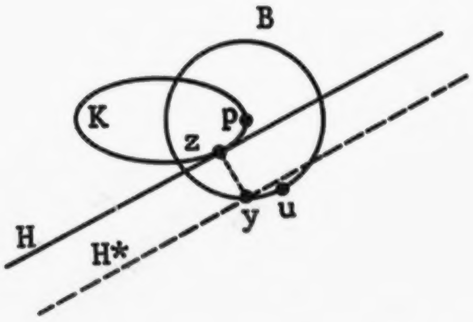
Let $A \subset E^d$, let $x \in E^d$ be a non-zero vector, and let $\lambda \in R$. The hyperplane $H(x,\lambda)$ is called a supporting hyperplane of A provided either

$\lambda = \sup\{\langle x,a\rangle \mid a \in A\}$ or $\lambda = \inf\{\langle x,a\rangle \mid a \in A\}$; in the first case we shall say that $H(x,\lambda)$ is the supporting hyperplane of A in direction x (or with outward normal x), and we shall denote it by $H(A;x)$.

From 9 we have:

10. If $K \subset E^d$ is convex, and if $C \subset \text{bd } K$ is convex (in particular, if C is a single point of bd K), there exists a supporting hyperplane H of K such that $C \subset H$.

The following independent proof of 10 can also be given when C is the single point p. Let B be a closed ball with center p and let y be a point in B which is farthest from the set K. Now $y \notin \text{cl } K$ since p must be in bd K, so there is a point $z \in \text{cl } K$ which is closest to y. Let H be the hyperplane normal to $[z,y]$ through z. As in previous arguments, y and K are separated by H, so if $z = p$, then H must be the desired supporting hyperplane and we are done. If $z \neq p$ then the hyperplane H^* parallel to H and through y must meet the interior of B. Thus some point $u \in B$ must be farther from K than y, a contradiction. (See Botts [1].)



A partial converse of 10 is

11. If $K \subset E^d$ is compact and convex, then each supporting hyperplane $H(y,\alpha)$ contains at least one point x such that $\langle x,y\rangle = \alpha$.

Note that compactness cannot be weakened to "closed" in 11; e.g., let $K = \{(a,b) \in E^2 \mid a > 0,\ b > 1/a\}$ and $H = \{(a,b) \mid b = 0\}$.

* * * *

There are two types of separation problems implicit in the above. We may ask how strong a separation is possible between two convex sets with given properties. Also we may assume certain conditions on one of the sets and require a certain type of separation, and ask what properties are necessary for the second set. In general, the stronger the given conditions on the first set, the weaker are the necessary conditions on the second set. See Klee [3] for theorems of this type.

It should also be noted that the separation of a flat and a convex set may be interpreted in a natural way as the Hahn-Banach theorem.

Exercises.

1. Show that for any convex $K \subset E^d$ we have rel bd $K = \emptyset$ if and only if $K = \text{aff } K$. Is the assumption that K be convex necessary?
2. Prove 2 and 3.
3. If K is not convex, is it possible that all the sets mentioned in 3 be different? Generalize, by taking repeated closures and relative interiors.
4. Prove that K_1 and K_2 may be [strictly] separated by a hyperplane if and only if it is possible to [strictly] separate 0 and $K_1 - K_2$.
5. Show that 5 remains valid if the assumptions on the convex sets $K_1, K_2 \subset E^d$ are weakened to either

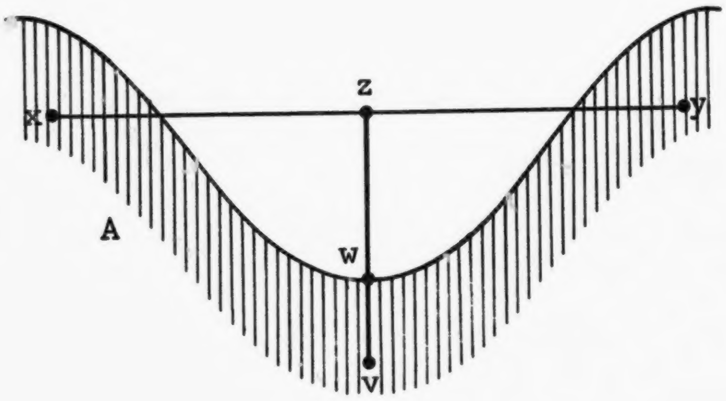
   (i) $\delta(K_1,K_2) > 0$;

   or (ii) K_1 is bounded and $\text{cl } K_1 \cap \text{cl } K_2 = \emptyset$.

6. Prove 7, by considering intersections of K_1 and K_2 with a sequence of balls with radii tending to infinity, and using 6.

7. Find examples in which $K \subset E^d$ is a closed convex set and H is a supporting hyperplane of K, but $H \cap K = \emptyset$. Show that for every bounded set A and every supporting hyperplane H of A we have $H \cap \text{cl}\, A \neq \emptyset$.

8. Prove the following converse of 10. (Compare the figure.) If A is a closed subset of E^d with $\text{int}\, A \neq \emptyset$, and if for each $w \in \text{bd}\, A$ there exists a supporting hyperplane H of A such that $w \in H$, then A is convex.



$x, y \in A$, $z \in A$, $v \in \text{int}\, A$, $w \in \text{bd}\, A$

9. Characterize those convex subsets of E^d which have no supporting hyperplanes.

10. Prove that each compact convex set in E^d is the intersection of closed balls.

11. If K_1 and K_2 are disjoint compact convex sets, then the set of all pairs (u,α) such that $H(u,\alpha)$ strictly separates K_1 and K_2, is an open subset of $S^{d-1} \times R$.

12. Let A denote a non-empty subset of E^d. The supporting function $h(x,A)$ of A is defined for all $x \in E^d$ by

$$h(x,A) = \sup\{\langle y,x\rangle \mid y \in A\}.$$

If for some non-zero $x \in E^d$ we have $h(x,A) < \infty$ then $H(A;x) = \{z \in E^d \mid \langle z,x\rangle = h(x,A)\}$ is a supporting hyperplane of A in direction x.

Prove:

(i) The supporting function $h(x,A)$ is positively homogeneous and convex, that is it satisfies

$$h(\lambda x,A) = \lambda h(x,A)$$

and

$$h(x+y,A) \leqq h(x,A) + h(y,A)$$

for all $\lambda \geqq 0$ and $x,y \in E^d$.

(ii) If $A \neq \emptyset$, $\lambda \geqq 0$, and $x,y \in E^d$ then

$$h(x,y+A) = \langle x,y\rangle + h(x,A)$$

$$h(x,\lambda A) = \lambda h(x,A)$$

$$h(x,\mathrm{cl}\, A) = h(x,A).$$

(iii) If A_1 and A_2 are non-empty and if $0 \neq x \in E^d$, then

$$H(A_1 + A_2;\ x) = H(A_1;\ x) + H(A_2;\ x)$$

$$(A_1 + A_2) \cap H(A_1 + A_2;\ x) = (A_1 \cap H(A_1;\ x)) \cap (A_2 \cap H(A_2;\ x)).$$

(iv) If $h(x)$ is any postively homogeneous and convex function on E^d such that $h(0) = 0$, there exists a unique non-empty closed convex set K such that

$$h(x) = h(x,K) \text{ for all } x \in E^d.$$

13. Find meaningful variants of the notions and results of this section relevant to spherically, or projectively, convex sets.
14. What becomes of the results of the present section if one considers convexity in the rational d-space?

Discussion.

Hausner suggested that a better name for "relative interior" would be "absolute interior," since it depends only on the given set and not the dimension of the space in which it is embedded. Perhaps it should be treated as the core of the set. Apparently the only justification for the use of the word "relative" is that the relative interior is just the usual interior in the affine subspace determined by the set when this space is given the relative topology.

Klee pointed out that the transfinite induction argument used in the second proof of 5 could be modified to a usual induction argument (thus being more palatable to undergraduates) by considering a countable dense subset of E^d rather than well-ordering all the points of E^d.

In a discussion of infinite-dimensional separation theorems it was observed that the algebraic and topological structures are not as closely related as they are in the finite dimensional case, and a part of the problem is that hyperplanes correspond to continuous linear functionals. The separation theorems are still valid in infinite dimensional space provided suitable hypotheses are added--indeed, they are major tools in the theory--but it is not true that two disjoint convex sets may be separated, as the following example shows: Let X be the (non-complete) vector space of all sequences $x = (x_1, x_2, \ldots)$ of real numbers, all but a finite number of which are zero. Let $K_1 \subset X$ be the subset of all sequences whose last non-zero number is positive. Let $K_2 = -K_1 \cup \{0\}$. Then K_1 and K_2 are convex disjoint subsets of X and $K_1 \cup K_2 = X$. Yet each K_i is topologically dense in X. Moreover, if

$$x = (x_1, \ldots, x_n, 0, 0, \ldots) \in K_1$$

(i.e., $x_n > 0$) and $y = (0, \ldots 0, -1, 0, 0, \ldots)$ with -1 in the $(n+1)$th place, then the halfray $\{x + ty \mid t > 0\} \subset K_2$ has $x \in K_1$ as endpoint. A similar statement is true for each $x \in K_2$. When presenting this example Klee gave the opinion that infinite-dimensional examples and results should be used to give a course broader scope whenever this may be easily done with only a little extra time.

Killgrove and Stratopoulos observed that a certain amount of algebra, topology and analysis seems to be a prerequisite for any undergraduate course using this material. After considerable discussion on the level at which a college geometry course should be taught and the amount of topology to be included, Kelly remarked that we are aiming here for an ideal course to be taught at good universities by a good staff. The universities have an important influence on the small colleges and courses designed for high school teachers. It is important that geometry courses should keep the respect of the profession and appear as an alive subject related to the rest of modern mathematics, since if it is thus accepted at the good schools, it will filter down to the weaker schools in perhaps a weaker form, but without the feeling that geometry is a dead subject that can well be left out of a crowded curriculum.

4. Convex hulls.

The space E^d is convex [and closed], and the intersection of any family of convex [and closed] sets is again convex [and closed]. Therefore the following definitions make sense:

The convex hull conv A of a subset A of E^d is the intersection of all the convex sets in E^d which contain A. The closed convex hull cl conv A of $A \subset E^d$ is the intersection of all the closed convex subsets of E^d which contain A.

Clearly, if A is bounded, so are conv A and cl conv A.

An immediate consequence of the definitions is

1. For every $A \subset E^d$ we have cl(conv A) = cl conv A.

Proposition 8 from the preceding section implies

2. cl conv A is the intersection of all the closed halfspaces which contain A.

A useful representation of conv A is given by

3. The convex hull conv A of a non-empty set $A \subset E^d$ is the set of all points which may be represented as convex combinations of points of A; that is, points which can be written in the form $\Sigma_{i=1}^{n} \alpha_i x_i$, where $x_i \in A$, $\alpha_i \geqq 0$, $\Sigma_{i=1}^{n} \alpha_i = 1$, $n = 1,2,\ldots$.

In many applications the following result is very important.

4. If A is a compact subset of E^d then conv A is closed; in other words, for compact A we have cl conv A = conv A.

Using the results of the preceding section it is not hard to give a direct proof of 4, by induction on the dimension d. Since a much simpler proof results from Carathéodory's theorem, we defer the proof of 4 till we establish 5.

The following theorem is one on the basic results in convexity, and has important application in other fields.

5. (Carathéodory's theorem) If A is a subset of E^d, then each $x \in \text{conv } A$ is expressible in the form $x = \Sigma_{i=0}^{d} \alpha_i x_i$ where $x_i \in A$, $\alpha_i \geqq 0$, $\Sigma_{i=0}^{d} \alpha_i = 1$.

Proof. Let $x \in \text{conv } A$ be given; let $x = \Sigma_{i=0}^{p} \alpha_i x_i$ (with $x_i \in A$, $\alpha_i \geqq 0$, $\Sigma_{i=0}^{d} \alpha_i = 1$) be a representation of x as a convex combination of points of A, involving the smallest possible number $p+1$ of points of A. We shall prove Carathéodory's theorem by showing $p \leqq d$. Indeed, assuming $p \geqq d + 1$ it follows that the set $(x_0, \ldots, x_p)$ is affinely dependent. Thus there exist β_i, $0 \leqq i \leqq p$, not all equal to 0, such that $\Sigma_{i=0}^{p} \beta_i x_i = 0$ and $\Sigma_{i=0}^{p} \beta_i = 0$. Without loss of generality we choose the notation so that $\beta_p > 0$ and $\frac{\alpha_p}{\beta_p} \leqq \frac{\alpha_i}{\beta_i}$ for all those i $(0 \leqq i \leqq p-1)$ for which $\beta_i > 0$. For $0 \leqq i \leqq p-1$, let $\gamma_i = \alpha_i - \frac{\alpha_p}{\beta_p} \beta_i$. Then $\Sigma_{i=0}^{p-1} \gamma_i = \Sigma_{i=0}^{p} \alpha_i - \frac{\alpha_p}{\beta_p} \Sigma_{i=0}^{p} \beta_i = 1$. Moreover, $\gamma_i \geqq 0$; indeed, if $\beta_i \leqq 0$ then $\gamma_i \geqq \alpha_i \geqq 0$; if $\beta_i > 0$ then $\gamma_i = \beta_i \left(\frac{\alpha_i}{\beta_i} - \frac{\alpha_p}{\beta_p}\right) \geqq 0$. Thus $\Sigma_{i=0}^{p-1} \gamma_i x_i = \Sigma_{i=0}^{p-1} \left(\alpha_i - \frac{\alpha_p}{\beta_p} \beta_i\right) x_i = \Sigma_{i=0}^{p} \alpha_i x_i = x$ is a representation of x as a convex combination of less than $p+1$ points of A, contradicting the assumed minimality of p. This completes the proof of Carathéodory's theorem.

The proof of 4 is now immediate. Indeed, if $x \in \text{cl conv } A$, there exists a sequence $x_n \in \text{conv } A$ such that $x = \lim_{n \to \infty} x_n$. By Carathéodory's theorem each $x_n = \Sigma_{i=0}^{d} \lambda_{n,i} x_{n,i}$, where $x_{n,i} \in A$, $0 \leqq \lambda_{n,i} \leqq 1$ for each n. The compactness of $[0,1]$ and of A guarantees the existence of converging subsequences $(\lambda_{n_k,i})$ and $(x_{n_k,i})$ such that $\lim_{k \to \infty} \lambda_{n_k,i} = \lambda^{(i)}$

and $\lim_{k\to\infty} x_{n_k,i} = x^{(i)}$. Then obviously $0 \leqq \lambda^{(i)} \leqq 1$, $\Sigma_{i=0}^{d} \lambda^{(i)} = 1$, $x^{(i)} \in A$, and $x = \Sigma_{i=0}^{d} \lambda^{(i)} x^{(i)}$, as claimed.

A result closely related to Carathéodory's, in the sense that either is easily derivable from the other, is Radon's theorem:

6. If A is a $(d+2)$-pointed subset of E^d, it is possible to find disjoint subsets A^*, A^{**} of A such that $\operatorname{conv} A^* \cap \operatorname{conv} A^{**} \neq \emptyset$.

A direct proof of Radon's theorem is very easy. Let $A = (x_0, \ldots, x_{d+1})$. Since $d+2$ points in d-space are affinely dependent there exist α_i, not all equal 0, such that $\Sigma_{i=0}^{d+1} \alpha_i = 0$ and $\Sigma_{i=0}^{d+1} \alpha_i x_i = 0$. Without loss of generality we choose the notation so that $\alpha_0, \ldots, \alpha_p$ are positive and $\alpha_{p+1}, \ldots, \alpha_{d+1}$ non-positive. Then $0 \leqq p \leqq d$. Let $\alpha = \Sigma_{i=0}^{p} \alpha_i > 0$ and define $\beta_i = \frac{\alpha_i}{\alpha}$ for $0 \leqq i \leqq p$ and $\gamma_i = -\frac{\alpha_i}{\alpha}$ for $p+1 \leqq i \leqq d+1$. The affine dependence of A can be rewritten in the form $\Sigma_{i=0}^{p} \beta_i x_i = \Sigma_{i=p+1}^{d+1} \gamma_i x_i$. Since $\beta_i \geqq 0$, $\gamma_i \geqq 0$, and $\Sigma_{i=0}^{p} \beta_i = \Sigma_{i=p+1}^{d+1} \gamma_i = 1$, this relation expresses $\operatorname{conv}(x_0, \ldots, x_p) \cap \operatorname{conv}(x_{p+1}, \ldots, x_{d+1}) \neq \emptyset$, as claimed by Radon's theorem.

Exercises.

1. Show that a hyperplane $H \subset E^d$ supports a set $A \subset E^d$ if and only if H supports $\operatorname{conv} A$.

2. Proposition 4 states that the convex hull of a compact set is compact; show that the convex hull of an open set is open. The convex hull of a closed set is not necessarily closed; find a closed set $A \neq \emptyset$ such that $\operatorname{conv} A$ is an open proper subset of the whole space.

3. For $A \subset E^d$ let $T(A) = \{\frac{1}{2}(x_1+x_2) \mid x_1, x_2 \in A\}$; let $T^1(A) = T(A)$,

and for $n \geqq 1$ let $T^{n+1}(A) = T(T^n(A))$. Denote $T^*(A) = \bigcup_{n\geqq 1} T^n(A)$. Show that $\text{cl } T^*(A) = \text{cl conv } A$, although in general $T^*(A) \neq \text{conv } A$. If $A = \text{bd } K$ where K is a bounded convex set in E^d, $d \geqq 2$, show that $\text{cl } K = T(A)$.

4. For $A \subset E^d$ let $\theta(A) = \{\lambda x_1 + (1-\lambda)x_2 \mid x_1, x_2 \in A,\ 0 \leqq \lambda \leqq 1\}$. Define $\theta^1(A) = \theta(A)$ and $\theta^{n+1}(A) = \theta(\theta^n(A))$ for $n \geqq 1$. Show that $\text{conv } A = \bigcup_{n\geqq 1} \theta^n(A)$. Characterize those convex sets $K \subset E^d$ for which $K = \theta(\text{bd } K)$. See Bonnice - Klee [1], Section 1.

5. Prove Steinitz's theorem: If $x \in \text{int conv } A \subset E^d$, there exists a subset A^* of A, containing at most $2d$ points such that $x \in \text{int conv } A^*$. Show that the number $2d$ may not be decreased in general, and characterize those A and x for which $2d$ points are needed in A.

6. Let $A \subset E^d$ be a finite set. Then $x \in \text{rel int conv } A$ if and only if x is representable as a convex combination of all points of A, with all coefficients positive.

7. Show that in Radon's Theorem 6 the sets A^* and A^{**} are unique if and only if every $d+1$ points of A are affinely independent. Show also that in this case two points of A belong to the same set if and only if they are separated by the hyperplane determined by the remaining d points.

8. What becomes of the above results if convex sets in the d-sphere, the projective d-space, or the rational d-space are considered?

What are the analogues of the above results if the convex combinations are replaced by other types of combinations? (See Section 2, remark 2.)

Discussion.

Klee presented the following alternate proof of 4: Let $D^{d+1} = \{(\lambda_0,\ldots,\lambda_d) \in E^{d+1} \mid \lambda_i \geqq 0, \Sigma \lambda_i = 1\}$ which is clearly compact. Let X denote the compact set

$$D^{d+1} \times \underbrace{A \times \ldots \times A}_{d+1 \text{ times}} .$$

Map X into E^d by $((\lambda_0,\ldots,\lambda_d), a_0,\ldots,a_d) \rightarrow \Sigma_{i=0}^{d} \lambda_i a_i$. By Carathéodory's theorem, the image of X is exactly conv A, and the map is clearly continuous. The continuous image conv A of the compact set X must be compact.

Grünbaum remarked that the original proof of Carathéodory used induction on the dimension d. This had the disadvantage of needing to know results about the intersection of compact sets with their supporting hyperplanes, while the present proof is more elementary.

Klee pointed out that in Carathéodory's theorem, if $p \in$ conv A and $x \in A$, then $p \in B$ for some set $B \subset A$ of at most d+1 points, one of which is x. Only one such point x may be thus specified in advance. A slight modification of the given proof gets this stronger result.

A similar generalization was reported recently by Motzkin [2]. Carathéodory's theorem has had a large number of uses in other fields as well as convexity; for example, use of Carathéodory's theorem and the separation theorems have significantly simplified proofs of some results about doubly stochastic matrices.

Johnson and Grünbaum mentioned that if the given set of d+2 points in Radon's theorem is not degenerate, specifically if the points are in general position, then there is a unique way to divide them into two disjoint subsets whose convex hulls meet.

Reay mentioned that if a set $A \subset E^d$ of $(d+1) + (k+1)$ points $(0 \leqq k \leqq d)$ is in general position, then it may be divided into two disjoint subsets whose convex hulls meet in a k-dimensional set. More generally, if A has $(r-1)(d+1) + (k+1)$ points, then it may be divided into r disjoint subsets whose convex hulls meet in a k-dimensional set. However, when $r > 2$, then an independence stronger than general position is necessary for the set A. See Reay [1].

5. Helly's Theorem.

The present section is devoted to a result known as Helly's theorem. It has many applications to different fields, and it occupies a central position among combinatorial-geometric problems.

1. (Helly) Let $\mathcal{K} = \{K_1,\dots,K_n\}$ be a finite family of convex sets in E^d, $d \geqq 1$. If for each subfamily $\mathcal{K}'$ of $\mathcal{K}$, consisting of at most $d+1$ sets, there exists a point common to all members of $\mathcal{K}'$, then there exists a point common to all sets of $\mathcal{K}$.

It is easy to give examples which show that the convexity of each set K_i is necessary. If $K_n \subset E^1$ is the closed ray $[n,\infty)$ for each integer n, all hypotheses hold except the finiteness of $\mathcal{K}$, yet the conclusion fails. So $\mathcal{K}$ must be finite. (See 2 below.)

We shall give two proofs of Helly's theorem; each of them exhibits certain features which reappear in the proofs of many combinatorial-geometric results. Note that no generality is lost in assuming $n \geqq d + 2$.

The first proof is due to Helly himself. It consists of two technically distinct steps.

(i) It is enough to prove the theorem under the additional assumption that each set is compact. Indeed for each $(d+1)$-tuple $\{i_0,\dots,i_d\} \subset \{1,\dots,n\}$ let $x(i_0,\dots,i_d) \in \cap_{j=0}^{d} K_{i_j}$. Let X_j, for $j = 1,\dots,n$, be defined by $X_j = \left\{x(i_0,\dots,i_d) \mid j \in \{i_0,\dots,i_d\}\right\}$, and let $C_j = \operatorname{conv} X_j$. Since X_j is a finite set, each C_j is compact; also, $X_j \subset K_j$ implies $C_j \subset K_j$. Moreover, the intersection of any $d+1$ sets $C_{j_0},\dots,C_{j_d}$ is non-empty, since it contains $x(j_0,\dots,j_d)$. But if Helly's theorem is assumed for families of compact convex sets, we have

$$\cap_{i=1}^{n} K_i \supset \cap_{i=1}^{n} C_i \neq \emptyset,$$

hence, the validity of Helly's theorem for general convex sets. Note that the finiteness of $\mathcal{K}$ is necessary in this part of the proof.

(ii) In order to prove Helly's theorem for families $\mathcal{K}$ consisting of compact convex sets we first note its validity for $d = 1$. Indeed, it is immediate that each K_i (which is, for $d = 1$, a closed segment, possibly degenerating to a point) contains the rightmost of the left endpoints of the members of $\mathcal{K}$. Assume now that the theorem is false; among the families for which it fails, choose one for which d is as small as possible; then $d > 1$. Among all the families in that E^d for which the theorem does not hold, select a family $\mathcal{K}$ with smallest possible cardinality n. Then $n > d + 1$, and for each subfamily of $\mathcal{K}$ containing $n-1$ members, the intersection of its members is non-empty. Let $K = \cap_{i=0}^{n-1} K_i \neq \emptyset$; then $K \cap K_n = \emptyset$ and by Theorem 3.5 there exists a hyperplane H strictly separating K and K_n. In particular, $K \cap H = \emptyset$. Let $C_i = K_i \cap H$ for $i = 1,\ldots,n-1$. Then each C_i is a compact convex subset of the $(d-1)$-dimensional space H; moreover, each $d = (d-1) + 1$ sets $C_{i_1},\ldots,C_{i_d}$ have a non-empty intersection: $\cap_{j=1}^{d} C_{i_j} = H \cap \left(\cap_{j=1}^{d} K_{i_j}\right) \neq \emptyset$ because $\cap_{j=1}^{d} K_{i_j}$ has points in K_n as well as in K. Hence, by the minimality assumption on $\mathcal{K}$ and the choice of H we have the contradiction $\emptyset \neq \cap_{i=0}^{n-1} C_i = H \cap \left(\cap_{i=1}^{n-1} K_i\right) = H \cap K = \emptyset$. This completes the first proof of Helly's theorem.

The second proof of Helly's theorem goes back to Radon. We began by establishing its validity in case $n = d + 2$. Simplifying the notation used in (i) above, let, for $i = 1, 2,\ldots, n = d + 2$,

$$x_i \in \bigcap_{\substack{j \neq i \\ 1 \leqq j \leqq n}} K_j .$$

By Radon's Theorem 4.6 there exist sets $J_1, J_2 \subset \{1,\dots,n\}$ such that $J_1 \cap J_2 = \emptyset$, $J_1 \cup J_2 = \{1,\dots,n\}$, and

$$\operatorname{conv}\{x_i \mid i \in J_1\} \cap \operatorname{conv}\{x_i \mid i \in J_2\} \neq \emptyset .$$

Let x be any point of this intersection; then $x \in K_j$ for each $j \in J_2$ since each such K_j contains all x_i with $i \in J_1$ and hence K_j contains also their convex hull. Reversing the roles of J_1 and J_2 we see that $x \in K_j$ also for all $j \in J_1$, i.e., $x \in \bigcap_{i=1}^{n} K_i \neq \emptyset$.

Hence Helly's theorem is valid for all families in E^d consisting of $n \leqq d+2$ sets. Now we apply induction on n. From a family $\mathcal{K} = \{K_1,\dots,K_n\}$, with $n > d+2$, we form a family $\mathcal{C} = \{C_1,\dots,C_{n-1}\}$, where $C_i = K_i \cap K_n$ for $i=1,\dots,n-1$. The intersection of each $d+1$ members of $\mathcal{C}$ is non-empty since it coincides with the intersection of some $d+2$ members of $\mathcal{K}$, which is non-empty by the above special case. But the inductive assumption then implies $\emptyset \neq \bigcap_{i=1}^{n-1} C_i = \bigcap_{j=1}^{n} K_j$, and the proof is completed.

Using the standard compactness arguments, it is easy to derive from 1 the following infinite version of Helly's theorem.

2. If $\mathcal{K}$ is any family of compact convex subsets of E^d such that every $d+1$ or fewer members of $\mathcal{K}$ have a non-empty intersection, then the intersection of all members of $\mathcal{K}$ is non-empty.

An extensive list of results related to Helly's theorem, with references to the original papers, may be found in the Danzer-Grünbaum-Klee survey listed in the bibliography. The following exercises are meant only to give the reader the opportunity to try his hand at applying either one of the above versions of Helly's theorem, or the techniques used in their proofs.

Exercises.

1. Theorem 2 remains valid if the convex sets K_i are assumed to be closed, provided some finite subfamily has a bounded intersection. (Even the "finite" of the preceding sentence may be omitted, but then some additional facts--e.g., Theorem 7.3--have to be used.)

2. Let $\mathcal{K}$ be a family of compact, convex subsets of E^d such that, for some fixed k with $1 \leqq k \leqq d+1$, each k or fewer members of $\mathcal{K}$ have a common point. Then for every flat $L \subset E^d$ of dimension $d + 1 - k$ there exists a flat L' of the same dimension, parallel to L which intersects all the members of $\mathcal{K}$.

3. Let $\mathcal{K}$ be a family of compact convex sets in E^d and let $C \subset E^d$ be a compact convex set. If for every $d+1$ members of $\mathcal{K}$ there exists a translate of C intersecting all of them, then there exists a translate of C which intersects all the members of $\mathcal{K}$. A valid statement results also if "intersecting" is replaced throughout by "containing," or by "contained in."

A special case of this result was first established by Vincensini [1]. The following simple proof is due to Klee [2].

For each $K_i \in \mathcal{K}$, let $C_i = \{x \in E^d \mid (x + C) \cap K_i \neq \emptyset\}$. Each C_i is convex, and each $d+1$ or fewer of the C_i have a point in common. By Helly's theorem, there is a point x common to all C_i, so $(x + C) \cap K_i \neq 0$ for each $K_i \in \mathcal{K}$.

4. Let A and B be two finite sets of points in E^d. Then A and B may be strictly separated by a hyperplane if and only if for each $X \subset A \cup B$ such that X contains at most $d + 2$ points, there exists a hyperplane strictly separating $X \cap A$ from $X \cap B$.

5. Let $\mathcal{L}$ be a finite family of parallel line segments in E^2, each three of which admit a common line transversal (i.e., a line which intersects each of them). Then there is a line transversal common to all members of $\mathcal{L}$.

Assume with no loss of generality that $\mathcal{L}$ has at least three members and all segments are parallel to the Y-axis. For each segment $L \in \mathcal{L}$ let C_L denote the set of all points $(a,b) \in E^2$ such that L is intersected by the line $y = ax + b$. Each C_L is convex and each three such sets have a common point, so by Helly's theorem there is a point $(a_0,b_0) \in \bigcap_{L \in \mathcal{L}} C_L$. The line $y = a_0x + b_0$ is a common transversal.

Note that this may be extended in several ways. Using the first exercise above, if all the line segments are closed, then the finiteness restriction of $\mathcal{L}$ may be relaxed. Instead of using transversal lines, we may use transversal polynomials of degree n, provided we demand that each $(n+2)$ of the given segments have a common n-polynomial as transversal.

Also see Rademacher and Schoenberg [1] for other applications of Helly's theorem.

The general idea of applying Helly's theorem is to prove something for small sets ($d+1$ or fewer elements) of certain objects in order to conclude that it is true for the set of all the objects. It has frequently happened that the collection of objects considered in a particular problem was too small to be suitable for an inductive proof. For example, if we wished to prove Helly's theorem for the family of all elliptical domains in E^2, any attempt to work with the foci or other geometric properties of the ellipse would probably end in a disaster.

Discussion.

Klee mentioned that a negative form of Helly's theorem is frequently very useful, particularly in analysis: If $\mathcal{K}$ is a family of convex sets (with suitable restrictions) with empty intersection, then some small subfamily must already have empty intersection.

One other application of Helly's theorem is the following (see DGK [1] Theorem 2.5): Let X be an infinite compact set in E^d (a d-dimensional art gallery), and suppose that for each $d+1$ points of X there is a point $x \in X$ from which each of these $d+1$ points are visible (i.e., $[x,u_i] \subset X$). Then X is starshaped (i.e., there is some $x \in X$ from which each point of the art gallery is visible).

This problem leads directly to many unsolved "illumination" problems. A famous one mentioned by Klee was: Given a triangular billard table, is there some point and some direction so that a (one-pointed) billard ball will describe a path which is dense in the triangle? Is it true that from all points there exists a direction which gives such a path? For general triangles almost all such meaningful questions of this type are unsolved.

It was pointed out that various theorems similar to Exercise 5 are known where the segments of that exercise are replaced by various other kinds of sets. These results lead many (including Helly himself) to look for a version of Helly's theorem that was valid in a much more general setting. One such generalization was given for non-convex sets and is called Helly's topological theorem.

It was noted that Grünbaum has generalized Helly's theorem in the following way (see Grünbaum [1]): Let $\mathcal{K}$ be a finite family of convex sets in E^d, and suppose $0 \leq k \leq d$. If each $h(k)$ or fewer sets of $\mathcal{K}$ have an intersection which is at least k-dimensional, then $\cap \mathcal{K}$ is k-dimensional, where

$$h(0) = d + 1$$

$$h(k) = 2d + 1 - k \qquad (1 \leqq k \leqq d-1)$$

$$h(d) = d + 1$$

Note that this is just Helly's theorem if $k = 0$ and Vincensini's theorem if $k = d$.

6. Extreme and exposed points; faces and poonems.

Let K be a convex subset of E^d. A point $x \in K$ is an extreme point of K provided $y,z \in K$, $0 < \lambda < 1$, and $x = \lambda y + (1-\lambda)z$ imply $x = y = z$. In other words, x is an extreme point of K if it does not belong to the relative interior of any segment contained in K. The set of all extreme points of K is denoted by ext K. Clearly, if $x \in \text{ext } K$ then $x \notin \text{conv}(K \sim \{x\})$.

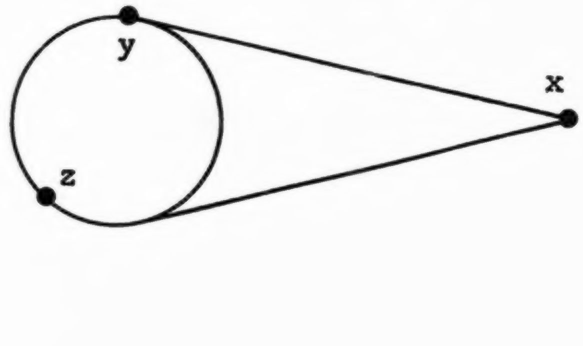
Let K be a convex subset of E^d. A set $F \subset K$ is a face of K if either $F = \emptyset$ or $F = K$, or if there exists a supporting hyperplane H of K such that $F = K \cap H$. $\emptyset$ and K are called the improper faces of K. The set of all faces of K is denoted by $\mathfrak{F}(K)$. A point $x \in K$ is an exposed point of K if the set $\{x\}$, consisting of the single point x, is a face of K. The set of all exposed points of K is denoted by exp K. If K is a closed convex set, it is obvious that each $F \in \mathfrak{F}(K)$ is closed. The notations ext K, exp K and $\mathfrak{F}(K)$ will in the sequel be used only for closed convex sets K. Note that the definition of exposed point involves the "outside of K," i.e., supporting hyperplanes, while the definition of extreme point involves the "inside of K," i.e., segments in K.

The following statements result at once from the definitions:

1. If $F \in \mathfrak{F}(K)$ and if $K' \subset K$ is a closed convex set, then $F \cap K' \in \mathfrak{F}(K')$.

2. If $F \in \mathfrak{F}(K)$ and if $x \in F$, then $x \in \text{ext } K$ if and only if $x \in \text{ext } F$; thus, if $F \in \mathfrak{F}(K)$, then $\text{ext } F = F \cap \text{ext } K$.

3. For every convex $K \subset E^d$ we have $\text{exp } K \subset \text{ext } K$. If $K \subset E^2$ is the convex hull of a circle and an exterior point x, then the point y of tangency is an extreme point, but not an exposed point, because the only face of K containing y is the



segment $[x,y]$. Note that z is an exposed point with a unique supporting hyperplane while x admits many supporting hyperplanes.

4. Let K be a closed convex set in E^d, let $x \in K$, and let B be a solid ball centered at x. Then $x \in \text{ext } K$ if and only if $x \in \text{ext}(K \cap B)$, while $x \in \exp K$ if and only if $x \in \exp(K \cap B)$.

The next two results explain the role of the extreme points.

5. Let K be a compact convex subset of E^d. Then $K = \text{conv}(\text{ext } K)$. Moreover, if $K = \text{conv } A$ then $A \supset \text{ext } K$.

Proof. Clearly, $K \supset \text{conv}(\text{ext } K)$. In order to establish $K \subset \text{conv}(\text{ext } K)$, we use induction on the dimension of the convex set K, the assertion being obvious in case $\dim K$ is -1, 0, or 1. Without loss of generality we assume $E^d = \text{aff } K$. Let $x \in K$. If $x \notin \text{ext } K$, let L be a line such that $x \in \text{rel int } (L \cap K)$. Then $L \cap K$ is a segment $[y,z]$, where obviously $y,z \in \text{bd } K$. Since through each boundary point of the convex set K there passes a supporting hyperplane, there exist faces F_y and F_z of K containing y and z respectively. Now, the dimensions of F_y and F_z are smaller than $\dim K$; by the inductive assumption, $F_y = \text{conv}(\text{ext } F_y)$ and $F_z = \text{conv}(\text{ext } F_z)$. Using statement 2 (above) we have $x \in \text{conv}(y,z) \subset \text{conv } (F_y \cup F_z) \subset \text{conv}(\text{conv}(\text{ext } F_y) \cup \text{conv}(\text{ext } F_z)) \subset \text{conv}(\text{ext } F_y \cup \text{ext } F_z) \subset \text{conv}(\text{ext } K)$, as claimed. The last assertion of the theorem being obvious, this completes the proof of 5.

An analogous inductive proof yields also

6. Let K be a closed convex subset of E^d, which contains no line. Then $\text{ext } K \neq \emptyset$.

Regarding exposed points, we have

7. Let $K \subset E^d$ be a compact set and let $H^+ = \{x \in E^d \mid \langle x,u \rangle > \alpha\}$

(where u is a unit vector) be an open halfspace such that $H^+ \cap K \neq \emptyset$. Then $H^+ \cap \exp K \neq \emptyset$.

Proof. Let $K^* = H^+ \cap K$, let $y \in K^*$, and denote by ε the distance from y to $\operatorname{bd} H^+$ and by δ the number $\delta = \max\{\rho(x,y-\varepsilon u) \mid x \in K \cap \operatorname{bd} H^+\}$. Let $z = y - \beta u$, where β is some fixed number satisfying $\beta > \frac{\delta^2 + \varepsilon^2}{2\varepsilon}$. Denoting by B the solid unit ball centered at the origin, let $\mu = \inf\{\lambda > 0 \mid z + \lambda B \supset K^*\}$. Clearly, $\mu \geqq \beta$. Then by the compactness of $\operatorname{cl} K^*$, we have $z + \mu B \supset K^*$ and $C = (\operatorname{cl} K^*) \cap \operatorname{bd}(z + \mu B) \neq \emptyset$. We claim that $C \cap \operatorname{bd} H^+ = \emptyset$. Indeed, assuming the existence of a point $v \in C \cap \operatorname{bd} H^+$, we would have $\delta^2 \geqq (\rho(v,y-\varepsilon u))^2 = \mu^2 - (\beta-\varepsilon)^2 \geqq 2\beta\varepsilon - \varepsilon^2$ which implies $2\beta\varepsilon \leqq \delta^2 + \varepsilon^2$, in contradiction to the choice of β. Therefore, $C \subset K^*$ but clearly each point of C is an exposed point of $z + \mu B$ and therefore also of $\operatorname{cl} K^*$ and of K, as claimed.

Lemma 7, together with 4, 5, and 6 above, 8 from Section 3, and 4 from Section 4, imply Straszewicz' theorem:

8. If $K \subset E^d$ is a compact convex set then $\operatorname{cl} \operatorname{conv} \exp K = K$.

Indeed, let $K' = \operatorname{cl} \operatorname{conv} \exp K$; obviously $K' \subset K$. If $K' \neq K$, then there exists an $x \in K$ such that $x \notin K'$. Since the compact convex sets $\{x\}$ and K' may be strictly separated, there exists an open halfspace H^+ such that $H^+ \cap K \neq \emptyset$. But then $H^+ \cap \exp K \neq \emptyset$ by Theorem 7, contradicting the definition of K'.

The reader is invited to prove

9. If $K \subset E^d$ is a closed convex set, then $\exp K \subset \operatorname{ext} K \subset \operatorname{cl} \exp K$; therefore, if K contains no line, then $\exp K \neq \emptyset$. [Hint: K closed, bounded and convex in E^d implies K is compact. (Note that this may not be true in more general settings.) Hence $K = \operatorname{cl} \operatorname{conv} \exp K \subset \operatorname{cl} \operatorname{conv} \operatorname{cl} \exp K =$

$\operatorname{conv} \operatorname{cl} \exp K = K$. Thus by the second part of Theorem 5, $\operatorname{ext} K \subset \operatorname{cl} \exp K$.]

To get an example in E^3 of a set K for which $\exp K$ is not closed, consider $\operatorname{conv}(C \cup S)$ where C is a circle and S is a closed line segment, not in the plane of C, whose relative interior meets C.

Regarding the family $\mathfrak{F}(K)$ of all faces of a closed convex set K we have:

10. The intersection $F = \bigcap_{i=1}^{r} F_i$ of any family $\{F_i\}$ of faces of a closed convex set K is itself a face of K.

Proof. If $F = \emptyset$ the assertion is true according to our definitions; thus we shall consider only the case $F \neq \emptyset$. Without loss of generality, we may assume that the origin 0 belongs to F, and that each F_i is a proper face of K. Then the face F_i is given by $F_i = K \cap \{x \mid \langle x,u_i\rangle = 0\}$ where u_i is some unit vector such that $K \subset \{x \mid \langle x,u_i\rangle \geqq 0\}$. Let $H = \{x \mid \langle x,v\rangle = 0\}$ where $v = \Sigma_{i=1}^{r} u_i$; then clearly $K \subset \{x \mid \langle x,v\rangle \geqq 0\}$. Since $0 \in K \cap H$, this implies that H is a supporting hyperplane of K. Now, if $x \in F$, then $\langle x,u_i\rangle = 0$ for all i and therefore $\langle x,v\rangle = 0$; hence $x \in H \cap K$ and thus $F \subset H \cap K$. On the other hand, if $x \in K \sim F$, then $\langle x,u_j\rangle > 0$ for at least one j and $\langle x,v\rangle \geqq \langle x,u_j\rangle > 0$; thus $x \notin H \cap K$. Therefore, $F = H \cap K$ and F is a face of K, as claimed.

The family $\{F_i\}$ in Theorem 10 may be infinite; in this case the face of smallest dimension obtainable as an intersection of finite subfamilies of $\{F_i\}$ equals the intersection of all members of $\{F_i\}$.

It is easy to find examples which show that the following situation is possible: K is a compact convex set, $C \in \mathfrak{F}(K)$ and $F \in \mathfrak{F}(C)$, but $F \notin \mathfrak{F}(K)$. Consider the example in 3 above; $F = \{y\}$ and $C = [x,y]$.

This observation leads to the following definition:

A set F is called a poonem* of the closed convex set K provided there exist sets $F_0,\dots,F_k$ such that $F_0 = F$, $F_k = K$, and $F_{i-1} \in \mathfrak{F}(F_i)$ for $i = 1,\dots,k$. Alternatively, a set P is a poonem of K provided $P = K \cap \operatorname{aff} P$ and $K \sim P$ is convex.

By this definition, each face of K is also a poonem of K but the converse is not true in general. However, each $(d-1)$-dimensional poonem of $K \subset E^d$ is a face (facet) of K, so the facets and $(d-1)$-poonems coincide. Clearly, each poonem F is a closed convex set, and $\operatorname{ext} F = F \cap \operatorname{ext} K$. Thus each extreme point of K is a poonem and conversely, each 0-dimensional poonem is an extreme point. One of the characteristics of poonems that makes them a valuable tool is just that fact, namely, the poonems are related to extreme points analogously to the way faces are related to exposed points of K. The set of all poonems of a closed convex set K shall be denoted by $\mathcal{P}(K)$.

The reader is invited to deduce from Theorem 10 the analogous result:

11. The intersection $F = \bigcap_i F_i$ of a family $\{F_i\}$ of poonems of a closed convex set K is in $\mathcal{P}(K)$.

12. If $F \in \mathcal{P}(K)$, then $\mathcal{P}(F) = \{P \in \mathcal{P}(K) \mid P \subset F\}$.

13. If $F \in \mathfrak{F}(K)$ and $P \in \mathcal{P}(K)$, then $P \cap F \in \mathcal{P}(F)$ and $P \cap F \in \mathfrak{F}(P)$.

Exercises.

1. A convex cone has at most one exposed point.

2. Let K denote a compact convex set. Show that if $\dim K \leq 2$, then $\operatorname{ext} K$ is closed, but $\exp K$ is not necessarily closed. Find a $K \subset E^3$ such

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* "Poonem" is derived from the Hebrew word for "face."

that $\exp K \neq \operatorname{ext} K \neq \operatorname{cl} \exp K$.

3. If the family $\mathcal{F}(K)$ of all faces of a closed convex set K is partially ordered by inclusion, then $\mathcal{F}(K)$ is a complete lattice. (For lattice-theoretic notions see, for example, the books of Birkhoff or Szász.) The same is true for the family $\mathcal{P}(K)$ of all poonems of K. (In both cases the greatest lower bound of a family of elements is their intersection.)

4. If K is a closed convex set and if C is a subset of K, show that $C \in \mathcal{P}(K)$ is equivalent to each of the following conditions:

(i) C is convex and for every pair x,y of points of K either the closed segment $[x,y]$ is contained in C, or else the open interval (x,y) does not meet C.

(ii) $C = K \cap \operatorname{aff} C$ and $K \sim \operatorname{aff} C$ is convex.

(iii) There is a flat L for which $C = K \cap L$ and $K \sim L$ is convex.

(iv) There exists an $x \in K$ such that C is the maximal convex subset of K satisfying $x \in \operatorname{rel\,int} C$.

5. If $F_i \in \mathcal{F}(K)$ for $0 \leqq i \leqq n$ and if $F_0 \subset \bigcup_{i=1}^{n} F_i$, then there exists i_0, $1 \leqq i_0 \leqq n$, such that $F_0 \subset F_{i_0}$. The same is true if all F_i belong to $\mathcal{P}(K)$.

6. Let K_1 and K_2 be closed convex sets. Prove:

(i) If $F_i \in \mathcal{F}(K_i)$ for $i = 1,2$, then $F_1 \cap F_2 \in \mathcal{F}(K_1 \cap K_2)$.

(ii) If $F_i \in \mathcal{P}(K)$ for $i = 1,2$, then $F_1 \cap F_2 \in \mathcal{P}(K_1 \cap K_2)$.

(iii) If $F \in \mathcal{P}(K_1 \cap K_2)$ there exist $F_1 \in \mathcal{P}(K_1)$ and $F_2 \in \mathcal{P}(K_2)$ such that $F = F_1 \cap F_2$.

(iv) If $\operatorname{rel\,int} K_1 \cap \operatorname{rel\,int} K_2 \neq \emptyset$ and if $F \in \mathcal{F}(K_1 \cap K_2)$, there exist $F_1 \in \mathcal{F}(K_1)$ and $F_2 \in \mathcal{F}(K_2)$ such that $F = F_1 \cap F_2$.

(v) Find examples showing that (iv) is not true if

$$\text{rel int } K_1 \cap \text{rel int } K_2 = \emptyset.$$

7. Let T be a non-singular projective transformation,

$$Tx = \frac{Ax + b}{\langle c,x\rangle + \delta},$$

and let H^+ be the open halfspace $H^+ = \{x \in E^d \mid \langle c,x\rangle + \delta > 0\}$. Prove:

(i) If A is any subset of H^+ then $T(\text{conv } A) = \text{conv } TA$.

(ii) For every compact convex set K for which T is permissible, $\mathfrak{F}(TK) = \{TF \mid F \in \mathfrak{F}(K)\}$ and $\mathcal{P}(TK) = \{TF \mid F \in \mathcal{P}(K)\}$.

Discussion.

Yale pointed out that the length of the chain $F_0,\ldots,F_k$ in the definition of poonem can be of length at most $d+1$ if $F_k = K \subset E^d$. On the other hand, in each space E^d there are examples of extreme points $\{x\} = F_0$ in a set $K = F_k$ for which the length of each such chain is exactly $d+1$. The example in 3 above shows this in E^2.

Klee remarked that contrary to what might be supposed, the relation between extreme points and exposed points in the boundary can be quite complex. For example, in E^3 there exist sets which have the following properties:

1) the non-extreme boundary points are dense in the boundary,

2) the extreme but not exposed points are dense in the boundary, and

3) the exposed points are dense in the boundary.

For a number of years it was thought that in some reasonable sense almost all extreme points should be exposed. In the 2-dimensional example in 3 above, there are only two extreme points which are not exposed. And in E^2 it is easy to see that there can be only countably many non-exposed extreme points, since each one would have to be the end of some line segment contained in the boundary of the convex body. Both Klee and Choquet tried to prove a higher dimensional version of this theorem and failed. A few years ago Corson [1] showed a 3-dimensional example of a convex body in which almost all (in a sense that can be made precise) of the extreme points fail to be exposed. For open problems in this area, see Choquet-Corson-Klee [1].

7. Unbounded convex sets.

The present section deals with some properties of unbounded convex sets.

1. A closed convex set $K \subset E^d$ is unbounded if and only if K contains a ray.

Proof. We shall consider only the non-trivial part of the assertion. Let $x_0 \in K$, and let $S = \text{bd } B$ denote the unit sphere of E^d centered at the origin. For each $\lambda > 0$ we consider the radial projection $P_\lambda = \pi(K \cap (x_0 + \lambda S))$ of the compact set $K \cap (x_0 + \lambda S)$ onto $x_0 + S$, the point x_0 serving as center of projection.* Since radial projection is obviously a homeomorphism between $x_0 + \lambda S$ and $x_0 + S$, the set P_λ is compact. If K is unbounded, then $P_\lambda \neq \emptyset$ for every $\lambda > 0$. Since K is convex and $x_0 \in K$, we have $P_\lambda \subset P_\mu$ whenever $\mu \leqq \lambda$. Therefore, $\bigcap_{\lambda>0} P_\lambda \neq \emptyset$. If y_0 is any point of this intersection, the ray $\{\lambda y_0 + (1-\lambda)x_0 \mid \lambda \geqq 0\}$ is clearly contained in K. This completes the proof of 1.

2. Let $K \subset E^d$ be closed and convex, let $L = \{\lambda z \mid \lambda \geqq 0\}$ be a ray emanating from the origin, and let $x,y \in K$. Then $x + L \subset K$ if and only if $y + L \subset K$.

Proof. Let $x + L \subset K$, and let $y + \lambda z \in y + L$ be given, $\lambda \geqq 0$. For $0 < \mu < 1$, consider the point $v_\mu = (1-\mu)y + \mu(x + \frac{\lambda z}{\mu}) \in K$. Since $\delta(v_\mu, y + \lambda z) = \delta(0, \mu(x-y))$, the distance between $y + \lambda z$ and v_μ is arbitrarily small provided $\mu > 0$ is sufficiently small. But K is closed, and therefore $v_\mu \in K$ implies $y + \lambda z \in K$. Since x and y play equivalent roles, the proof of 2 is completed.

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* If $x_0 \in E^d$, the radial projection π, with center of projection x_0, of $E^d \sim \{x_0\}$ onto the unit sphere x_0+S is defined by $\pi(x+x_0) = x_0 + \frac{x}{\|x\|}$.

A closed convex set $C \subset E^d$ is a cone with apex x_0 provided $-x_0 + C$ is a cone with apex 0. A cone C with apex x_0 is pointed provided $x_0 \in \text{ext}\, C$. Let C be a cone with apex 0. The following assertions are easily verified:

(i) The apices of C form a linear subspace $C \cap (-C)$ of E^d. Therefore, either C is pointed, or there exists a line, all points of which are apices of C.

(ii) $C = C + C = \lambda C$ for every $\lambda > 0$.

Conversely, if a non-empty closed set $C \subset E^d$ has property (ii), then C is a cone with apex 0.

The intersection of any family of cones with common apex x_0 is a cone with apex x_0. Therefore it is possible to define the cone with apex x_0 spanned by a set $A \subset E^d$ as the intersection of all cones in E^d which have apex x_0 and contain A. Though this notion is rather important in various investigations, we shall be more interested in another construction of cones from convex sets.

Let K be a convex set and let $x \in K$. We define

$$cc_x K = \{y \mid x + \lambda y \in K \text{ for all } \lambda \geqq 0\}.$$

Clearly, $cc_x K$ is a convex cone which has the origin as an apex. Lemma 2 implies that for closed K we have $cc_x K = cc_y K$ for all $x,y \in K$. Thus the subscript x is unnecessary and may be omitted. The convex cone ccK is called the characteristic cone of K. Using Lemmas 1 and 2, we obtain the following result:

3. If $K \subset E^d$ is a closed convex set, then ccK is a closed convex cone; moreover, $\mathrm{cc}K \neq \{0\}$ if and only if K is unbounded.

A closed convex set K shall be called line-free provided no (straight) line is contained in K. Using this terminology, Theorem 6.6 and the last part of Theorem 6.9 may be formulated as: If K is line-free, then $\mathrm{ext}\, K \neq \emptyset \neq \mathrm{exp}\, K$. It is also clear that every line-free cone is pointed.

Returning to Lemma 2, we note that it immediately implies: If L is a linear subspace of E^d such that $x + L \subset K$ for some x, then $y + L \subset K$ for every $y \in K$. Therefore, the following decomposition theorem results:

4. If $K \subset E^d$ is a closed convex set, there exists a unique linear subspace $L \subset E^d$ of maximal dimension such that a translate of L is contained in K. Moreover, denoting by L^* any linear subspace of E^d complementary to L, we have $K = L + (K \cap L^*)$, where $K \cap L^*$ is a line-free set.

Some information on the structure of line-free sets is given in the following theorem.

5. Let $K \subset E^d$ be an unbounded, line-free, closed convex set. Then $K = P + \mathrm{cc}K$, where P is the union of all bounded poonems of K.

Proof. We use induction on the dimension of K, the assertion being obvious if $\dim K = 1$. If $\dim K > 1$ and if $x \in K$, let $y \in \mathrm{rel\ bd}\, K$ and $z \in \mathrm{cc}K$ be such that $x = y + z$. (Since K is line-free, such a choice is possible; indeed, for any $t \in \mathrm{cc}K$, $t \neq 0$, there exists a $\lambda > 0$ such that $x - \lambda t \in \mathrm{rel\ bd}\, K$.) Let F be any proper face of K such that $y \in F$. If F is bounded then $F \subset P$ and $x \in P + \mathrm{cc}K$. If F is not bounded, the inductive assumption and $\dim F < \dim K$ imply that $y = v + w$, where $w \in \mathrm{cc}F$ and v belongs to P', the union of the bounded poonems of F. Since $P' \subset P$, $\mathrm{cc}F \subset \mathrm{cc}K$, and ccK is convex, it follows that

$x = y + z = v + w + z \in P' + \mathrm{cc}K + \mathrm{cc}K \subset P + \mathrm{cc}K$. Since obviously $K \supset P + \mathrm{cc}K$, this completes the proof of Theorem 5.

Since for each bounded poonem F of K, we have $\mathrm{ext}\, F = F \cap \mathrm{ext}\, K$ and $F = \mathrm{conv}\, \mathrm{ext}\, F$, Theorem 5 implies:

6. Let $K \subset E^d$ be a line-free, closed convex set. Then $K = \mathrm{cc}K + \mathrm{conv}\, \mathrm{ext}\, K$.

Exercises.

1. Show that Lemma 1 is valid even without the assumption that K is closed.

2. If K is any convex set in E^d, show that $x,y \in \mathrm{rel\ int}\, K$ implies $\mathrm{cc}_x K = \mathrm{cc}_y K$. Moreover, for $x \in \mathrm{rel\ int}\, K$ the characteristic cone $\mathrm{cc}_x K$ is closed.

3. Show that the decomposition Theorem 4 holds also if K is a relatively open convex set.

4. Let $K \subset E^d$ be a closed convex set; then $\mathrm{cc}K$ is the maximal (with respect to inclusion) subset $T \subset E^d$ with the property: For every $x \in K$, $x + T \subset K$.

5. Let $K \subset E^d$ be a closed convex set; then $\mathrm{cc}K = \left\{x \in E^d \mid \langle x,u\rangle \geqq 0\right.$ for all such u that there exists an α with $\left.K \subset \{z \mid \langle x,u\rangle \geqq \alpha\}\right\}$.

6. Let $K \subset E^d$ be a closed convex set such that $0 \in \mathrm{rel\ int}\, K$. Prove that

$$\mathrm{cc}K = \bigcap_{n=1}^{\infty} \left(\frac{1}{n}K\right).$$

7. Using the notation of the decomposition Theorem 4, let L^{**} denote another linear subspace of E^d complementary to L. Show that $L^{**} \cap K$

is an affine image of $L^* \cap K$.

8. If $K \subset E^d$ is a line-free, closed, convex set, then there exists a hyperplane H such that $H \cap K$ is compact and $\dim K = 1 + \dim(H \cap K)$.

9. If $K \subset E^d$ is a closed pointed cone with apex x_0, there exists a hyperplane H such that $H \cap K$ is compact and K is the cone with apex x_0 spanned by $H \cap K$.

10. Prove the following results converse to Theorems 5 and 6.

(i) If K is an unbounded, line-free, closed convex set and if $K = C + P$, where C is a cone with apex 0, then P contains all bounded poonems of K.

(ii) If K is a line-free, closed convex set and if $K = C + P$, where C is a cone with apex 0 and $P \neq \emptyset$ is a closed, bounded, convex set, then $C = \mathrm{cc}K$ and $P \supset \mathrm{conv}(\mathrm{ext}\, K)$.

11. A convex set K is called reducible provided $K = \mathrm{conv}\ \mathrm{rel\ bd}\ K$. Prove the following results:

(i) If K is a closed convex set then K is the convex hull of the union of all irreducible members of $\mathcal{P}(K)$.

(ii) Each irreducible closed convex set is either a flat or a closed half-flat.

12. Show that each d-dimensional closed convex set is homeomorphic with one of the following $d+2$ sets: (i) a closed halfspace of E^d; (ii) the product $E^{d-k} \times B^k$ for some k with $0 \leqq k \leqq d$, where B^k denotes the k-dimensional (solid) unit ball.

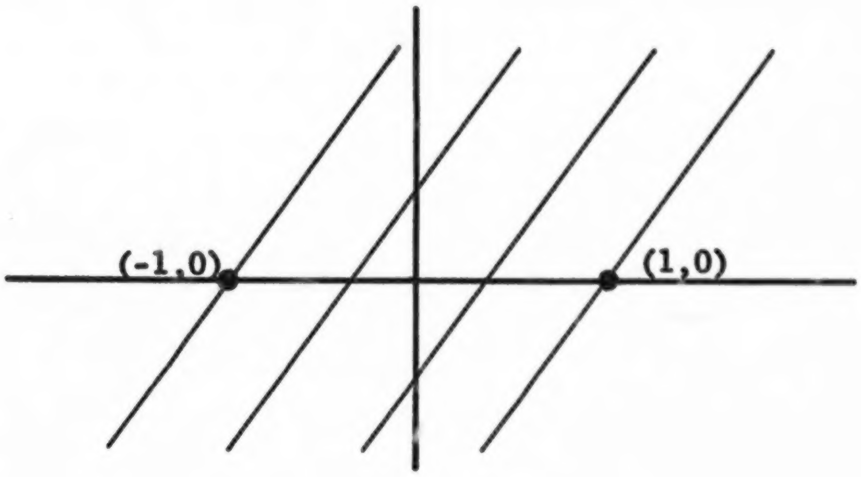
Discussion.

Since Sections 6, 7, and 8 were presented in the same lecture, many of the comments at the end of each section apply elsewhere as well.

There is no standard way of defining the "vertices" of an arbitrary closed convex set. If each exposed point is a vertex, then each boundary point of the circle would be a vertex. One definition could be: a vertex of $K \subset E^d$ is an exposed point which admits d independent supporting hyperplanes. Karlin and Shapley [1] have a discussion of degrees of exposure of points. A vertex to a set at a boundary point could be defined as a point which is the intersection of all supporting hyperplanes at the point.

Klee remarked that for a certain class of Banach algebras with unit, the unit itself is a vertex of the unit ball of the Banach algebra, and many useful results can be derived from this fact. See Bohnenblust and Karlin [1].

There are also simple examples in 2-dimensional rational space of closed convex sets which are linearly bounded but unbounded. This type of example can give the student a good feeling for the type of complication which in the real case arises only in the infinite dimensional spaces. This example also shows why "compact" is needed in many of these theorems rather than just "closed and bounded." In the rational plane consider all points between the "lines" through (-1,0) and (1,0) with a slope $\sqrt{2}$. Since all lines are



"rational," this unbounded set is linearly bounded. In this set we can get a sequence of decreasing nested closed non-empty sets whose intersection is empty.

Hausner pointed out that each d-dimensional set must have at least $d+1$ exposed points.

Koehler and Grünbaum pointed out that the best generalization of polytopes to countably-faceted bodies was the notion of quasi-polytopes, where the intersection with any polytope is a polytope. Demanding that P be the countable intersection of closed halfspaces is not a strong restriction, since each closed convex set $K \subset E^d$ is such a set. A wide variety of sets is also obtained by allowing finite intersections of either open or closed halfspaces.

Johnson and Prenowitz said it would be nice to get at the geometric content of these theorems. Most topological concepts, like compactness, can be expressed in geometric terms. In the rational example given above, a ray from an interior point does not have to intersect the boundary because of lack of completeness. Perhaps a geometric condition like "Any ray from an interior point meets the set of boundary points" should be imposed. Klee mentioned that in any finite-dimensional vector space over any ordered field, a line-free, segmentally-closed convex set is the convex hull of its extreme points together with its extreme rays. The more general question here appears to be "How far can you go with elementary geometrical concepts before you must impose further topological properties?"

8. Polyhedral sets.

A set $K \subset E^d$ is called a polyhedral set provided K is the intersection of a finite family of closed halfspaces in E^d.

Polyhedral sets have many properties which are not shared by all closed convex sets. One of the most important of these properties is:

1. Each poonem of a polyhedral set K is a face of K.

Before proving Theorem 1, we note a few facts about polyhedral sets.

Let $H_i^+ = \{x \in E^d \mid \langle x,u_i\rangle \geqq \alpha_i\}$, $1 \leqq i \leqq n$, be halfspaces, and let $K = \cap_{i=1}^n H_i^+$. Without loss of generality we shall in the present section assume that $\dim K = d$; we shall also say that a maximal proper face of K is a facet of K. The family $\{H_i^+ \mid 1 \leqq i \leqq n\}$ is called irredundant provided $K_i = \cap_{1\leqq j\leqq n}^{j\neq i} H_j^+ \neq K$ for each $i = 1, 2, \ldots, n$.

Denoting $H_i = \operatorname{bd} H_i^+ = \{x \in E^d \mid \langle x,u_i\rangle = \alpha\}$, we have

2. If $K = \cap_{j=1}^n H_j^+$, where $\{H_j^+ \mid 1 \leqq j \leqq n\}$ is irredundant, then $F_i = H_i \cap K$ is a facet of K.

This follows at once from the observation that $H_i \cap \operatorname{int} K_i \neq \emptyset$ which, in turn, is a reformulation of the irredundancy assumption. The same assumption also implies

3. $\operatorname{bd} K = \cup_{i=1}^n F_i$, where $F_i = H_i \cap K$ are the facets of K.

In particular, for each proper face F of K there exists a facet F_i of K such that $F \subset F_i$.

Let $F_i = H_i \cap K$ be a facet of K. Then

$$F_i = H_i \cap \left(\cap_{1\leqq j\leqq n}^{j\neq i} H_j^+\right) = \cap_{1\leqq j\leqq n}^{j\neq i} (H_i \cap H_j^+).$$

Thus F_i is a polyhedral set, namely the intersection of the sets $H_i \cap H_j^+$, $1 \leqq j \leqq n$, each of which is either H_i or a halfspace of the (d-1) dimensional space H_i. Therefore, by Theorem 3, each facet F of F_i is of the form $F = F_i \cap \text{rel bd } (H_i \cap H_j^+) = F_i \cap H_i \cap H_j = K \cap H_i \cap H_j = F_i \cap F_j$, for a suitable j. Thus:

4. A facet of a facet of a polyhedral set K is the intersection of two facets of K.

Now we are ready for the proof of the following result which, in view of Theorem 6.10, clearly implies Theorem 1.

5. Every poonem F of a polyhedral set K is an intersection of facets of K.

Proof. We shall use induction on $\dim K$, the assertion being obvious if $\dim K = 1$. If $\dim K > 1$, let $x \in \text{rel int } F$. By Theorem 3, there exists a facet F_i of K such that $x \in F_i$, i.e., $F \subset F_i$. Theorem 6.12 then implies that F is a poonem of F_i. Using the inductive assumption we see that F is an intersection of facets of F_i. Since each facet of F_i is the intersection of two facets of K, this completes the proof of Theorem 5.

We mention also the following immediate consequence of Theorem 5.

6. If K is a polyhedral set, then the family $\mathfrak{F}(K)$ is finite.

Exercises.

1. (See Theorem 7.4 for the notation.) Show that if K is a polyhedral set,

$$K = \cap_{i=1}^{n} \{x \in E^d \mid \langle x,u_i \rangle \geqq \alpha_i\},$$

then

$$L = \cap_{i=1}^{n} \{x \in E^d \mid \langle x,u_i \rangle = 0\}.$$

2. Show that if K is as above, then

$$\mathrm{cc}\ K = \bigcap_{i=1}^{n} \{x \in E^d \mid \langle x,u_i \rangle \geqq 0\}.$$

3. Show that if K is as above, and if $p \in K$ satisfies $\langle p,u_i \rangle = \alpha_i$ for $1 \leqq i \leqq m$, and $\langle p,u_i \rangle > \alpha_i$ for $m < i \leqq n$, then

$$\mathrm{cone}_p\ K = \bigcap_{i=1}^{m} \{x \in E^d \mid \langle x,u_i \rangle \geqq \alpha_i\}.$$

4. Show that every affine map of a polyhedral set is a polyhedral set, and that $K_1 + K_2$ is a polyhedral set provided K_1 and K_2 are polyhedral. Find a polyhedral set K and a projective transformation T (permissible for K) such that TK is not a polyhedral set.

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1. Helly's theorem on convex domains and Tchebycheffs' approximation problem. *Canadian Journal of Mathematics*, 2(1950), 245-256.

J. R. Reay

1. An extension of Radon's theorem. (To appear.)

P. Vincensini

1. Sur une extension d'un théorème de M. J. Radon sur les ensembles de corps convexes. *Bulletin de la Société Mathématique de France*, 67(1939), 115-119.

For more detailed bibliographic references of the topics covered in Part I, see the Danzer-Grünbaum-Klee paper (in the above list) and the book *Convex*

Polytopes of Grünbaum (listed below).

The material contained in Part I appears in various books on convexity. A rather complete list of the available books dealing with convexity and related subjects is given below.

Books on convexity and related topics

1. A. D. Aleksandrov (Alexandroff), *Convex polyhedra*. Moscow 1950 (Russian). (German translation: *Konvexe polyeder*, Berlin 1958.)

2. A. D. Aleksandrov (Alexandroff), *Intrinsic geometry of convex surfaces*, Moscow 1948 (Russian). (German translation: *Die innere Geometrie der konvexen Flächen*, Berlin 1955.)

3. R. V. Benson, *Euclidean geometry and convexity*. New York, McGraw-Hill, 1966.

4. W. Blaschke, *Kreis und Kugel*. Leipzig 1916. (Reprint: New York, Chelsea, 1949.)

5. V. G. Boltyanskiĭ and I. C. Gohberg, *Theorems and problems of combinatorial geometry*. Moscow 1965 (Russian).

6. T. Bonnesen and W. Fenchel, *Theorie der konvexen Körper*. Berlin 1934. (Reprint: New York, Chelsea, 1948.)

7. H. Busemann, *Convex surfaces*. New York, Interscience, 1958.

8. H. S. M. Coxeter, *Regular polytopes*. London, Macmillan, 1948. (Second edition, New York, 1963.)

9. H. G. Eggleston, *Problems in Euclidean space: Application of Convexity*. New York, Pergamon, 1957.

10. H. G. Eggleston, *Convexity*. Cambridge University Press, 1958.

11. L. Fejes Tóth, *Lagerungen in der Ebene, auf der Kugel und im Raum*. Berlin, Springer, 1953.

12. L. Fejes Tóth, *Regular figures*. New York, Macmillan, 1964.

13. W. Fenchel, *Convex cone, sets and functions*. Lecture notes, Princeton, 1953.

14. B. Grünbaum, *Convex polytopes*. New York, Wiley, 1967.

15. H. Hadwiger, *Altes und Neues über konvexe Körper*. Basel, Birkhauser, 1955.

16. H. Hadwiger, *Vorlesungen über Inhalt, Oberfläche und Isoperimetrie*. Berlin, Springer, 1957.

17. H. Hadwiger, H. Debrunner, and V. Klee, *Combinatorial geometry in the plane*. New York, Holt, Rinehart and Winston, 1964.

18. V. Klee (editor) *Convexity*. *Proceedings of Symposia in Pure Mathematics*, Volume 7. Providence, American Mathematical Society, 1963.

19. H. Kuhn and A. W. Tucker (editors), *Linear inequalities and related systems*. Princeton, 1956.

20. L. A. Lyusternik, *Convex figures and polyhedra*. Boston, Heath, 1966.

21. J. R. Reay, *Generalizations of a theorem of Carathéodory*. Memoirs of the American Mathematical Society, No. 54 (1965).

22. C. A. Rogers, *Packing and covering*. Cambridge, 1964.

23. F. A. Valentine, *Convex sets*. New York, McGraw-Hill, 1964.

24. I. M. Yaglom and V. G. Boltyanskiĭ, *Convex figures*. New York, Holt, Rinehart and Winston, 1961.

25. *Proceedings of the 1965 Copenhagen Colloquium on Convexity*. Københavns Universitets Matematiske Institut, 1967.

Part II

The material of the second week of these lectures is a departure into a more specialized area than that of last week. The present material would not be suitable for a year-long course in undergraduate geometry, but it would be appropriate for a number of one-quarter courses or honors papers at the senior level. In the second part of these lectures I will try to present more ideas and results, and therefore fewer proofs than last week. Much of this material has relatively easy proofs provided the correct sequence of proving these results is found. In general, since we work with finite systems, less background in analysis and topology is required. A student can get by with only an algebra background.

Unfortunately, the terminology in this area has not been standardized. Throughout this part of the notes the (perhaps unbounded) intersection of a finite number of closed halfspaces will be called a *polyhedral set*. A *d-polytope* will always denote a bounded d-dimensional polyhedral set in E^d.

These definitions, as well as most of the other notions, results and proofs of Part II are presented in detail in the book Grünbaum [1].

10. Number of facets of polytopes.

As mentioned before, the timing problem for computer programs has led to the general question "Given a polytope in E^d with f facets, how many vertices may it have?" Until about five years ago the answer was unknown except for certain easy cases. Thanks to some new ideas by Klee [2] in 1962 the problem is now almost solved. The little residue of the problem that is still left, however, is very challenging and strange.

Rather than attack the problem as stated, we will use the dual procedure, namely, find the maximum number of facets of a polytope given its dimension and the number of its vertices. These are really equivalent problems because of the existence of duality among polytopes. Here d = dimension of the polytope P, v = number of its vertices, $\mu(v,d)$ = maximum number of facets of a polytope in E^d with v vertices. To avoid trivial cases (analogous to many problems with the empty set) let us from now on assume $d \geqq 2$. This keeps a result like "$\mu(v,d)$ is a strictly increasing function of v" from being false or ill-defined. We also let $f_k(P)$ = the number of k-faces of P, $-1 \leqq k \leqq d$, and if P is understood we write f_k. The first result is immediate.

1. $\mu(v,2) = v$
2. In E^3, $f_2 \leqq \mu(f_0,3) = 2f_0 - 4$.

For a proof of 2, we just combine the following facts.

(1) $f_0 - f_1 + f_2 = 2$ (Euler's equation)

(2) Assume without loss of generality that P is a simplicial polytope (i.e., all its facets are (d-1)-simplices = triangles)

(3) counting the incidences of edges and facets of simplicial 3-polytopes in two different ways, $2f_1 = 3f_2$.

From (1) and (3), $f_2 = 2f_0 - 4 = \mu(f_0,3)$.

(4) Note that for each $f_0 \geqq 4$ we can get examples which show that these upper bounds are assumed.

In higher dimensions we try to go through these same steps but in general their proofs are harder, and together they might not always get the result. Further they frequently lead to unsolved problems.

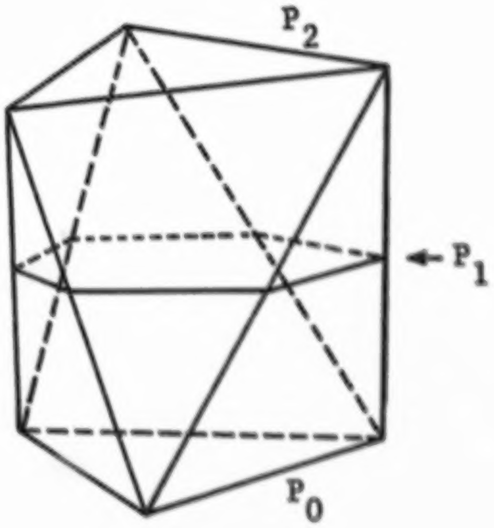
We first turn to a discussion of Euler's formula, the analogue of step (1) above.

3. For each d-polytope, $\Sigma_{i=-1}^{d} (-1)^i f_i = 0$.

Since $f_{-1} = f_d = 1$ for all d-polytopes, the formula frequently is given the form $\Sigma_{i=0}^{d-1} (-1)^i f_i = 1 - (-1)^d$. This formula was known to Schläfli as early as 1852 but the first published proofs of it came in the early 1880's when a number of independent proofs were published, all of which were incorrect (or rather incomplete) in their reasoning. (See Grünbaum [1], Chapter 8.) The basic idea in E^3 of each of these proofs was to start with one facet (and its associated edges and vertices) and note that $f_0 - f_1 + f_2 = 1$ holds. This formula remains valid if a second adjoining facet is added. These authors assumed that it is possible to always successively add facets of P (and their faces) in such a fashion that the newly added facet meets the already present 2-complex ((d-1)-complex) in a simply connected polygonal line segment ((d-2)-complex). This continues, preserving the formula $f_0 - f_1 + f_2 = 1$ (or $\Sigma_{i=0}^{d-1} (-1)^i f_i = 1$) until the last facet which must then be added individually, thus giving Euler's formula. The critical assumption of these proofs is certainly not obvious, even for $d = 3$. In case $d = 3$ the assumption may be established by an argument using the Jordan curve theorem. For $d \geqq 4$ the problem is still open and in fact, certain examples cast doubts on the validity of the assumption. There are simplicial complexes in E^3 for which the assumption is false, though none of these is known to correspond to a polytope. While Poincare seems to have given the first valid proof in 1899, the old proof continued to appear in books (e.g., Sommerville [2]). Most modern proofs used heavy topological machinery, and it was only in 1955 (Hadwiger [1]) and 1963 (Klee [1]) that elementary proofs were given, and even these had algebraic overtones. The following proof is completely elementary and stays within the framework of convex polytopes:

(1) Euler's equation obviously holds for $d = 1$ and $d = 2$, thus starting the induction on d.

(2) It holds if P is a prismoid, i.e., the convex hull of two polytopes P_0 and P_2, for which $(\text{aff } P_0) \cap (\text{aff } P_2) \cap P = \emptyset$.



To see this, let P_1 be the intersection of a hyperplane H with int P, where H is a hyperplane from the pencil of hyperplanes determined by aff P_0 and aff P_2. Then $f_0(P) = f_0(P_0) + f_0(P_2)$. Also if $1 \leqq k \leqq d-1$, a k-face of P is either a face of P_0 or P_2 or else it has vertices in both P_0 and P_2, in which case it corresponds to a (k-1)-face of P_1. Thus $f_k(P) = f_k(P_0) + f_k(P_2) + f_{k-1}(P_1)$ when $1 \leqq k \leqq d-1$. Summing over k we get

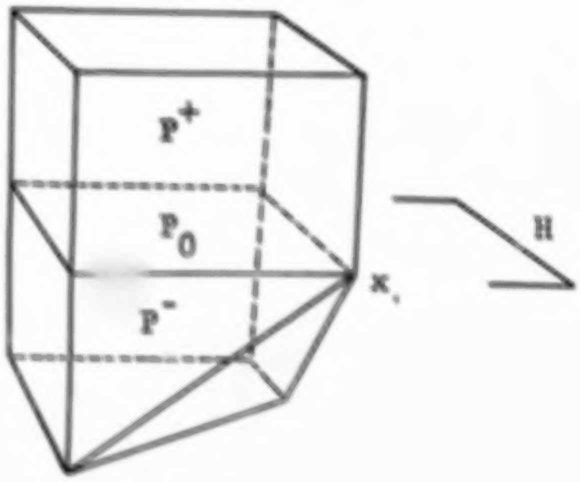
$$\Sigma_{k=0}^{d-1} (-1)^k f_k(P) = \Sigma_{k=0}^{d-1} (-1)^k (f_k(P_0) + f_k(P_2)) + \Sigma_{k=0}^{d-2} (-1)^{k+1} f_k(P_1).$$

By the induction applied to P_1, we see that Euler's formula holds:

$$\Sigma_{k=0}^{d-1} (-1)^k f_k(P) = 2 - (1 - (-1)^{d-1}) = 1 - (-1)^d.$$

(3) Euler's formula is similarly easy to deduce for any d-simplex, or any d-pyramid, i.e., the convex hull of a (d-1)-polytope and a point.

(4) Let P be any d-polytope, and let H be any hyperplane which meets int P and which contains exactly one vertex x of P. Let $P^+ = H^+ \cap P$ and $P^- = H^- \cap P$. If Euler's formula holds for the d-polytopes P^+ and P^- and for the (d-1)-polytope $P_0 = P^+ \cap P^- = H \cap P$, then it is valid for $P = P^+ \cup P^-$.



The proof is analogous to that of part (2) above when we notice that the following formulas hold:

$$f_0(P) = f_0(P^+) + f_0(P^-) - 2f_0(P_0) + 1$$

$$f_1(P) = f_1(P^+) + f_1(P^-) - 2f_1(P_0) - f_0(P_0) + 1$$

and if $2 \leq k \leq d-2$,

$$f_k(P) = f_k(P^+) + f_k(P^-) - 2f_k(P_0) - f_{k-1}(P_0).$$

(5) If P is any polytope in E^d, there is a hyperplane H such that no translate of H contains two or more vertices of P.

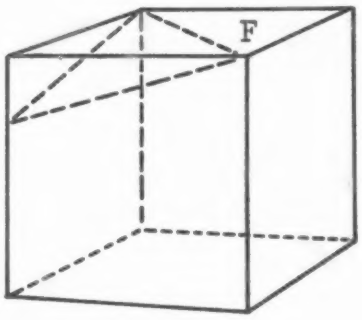
(6) Let $P \subset E^d$ be given, let $H_1, \ldots, H_v$ be parallel hyperplanes each of which meets P in exactly one vertex, and let $P_1, P_2, \ldots, P_{v-1}$ be the parts of P into which the H_i divide P. Note that P_1 and P_{v-1} are

d-pyramids, while $P_2, \ldots, P_{v-2}$ are prismoids. By (2) and (3) Euler's formula holds for each P_i. By repeated use of (4), Euler's formula holds for $P = \cup_{i=1}^{v-1} P_i$. This proves the theorem.

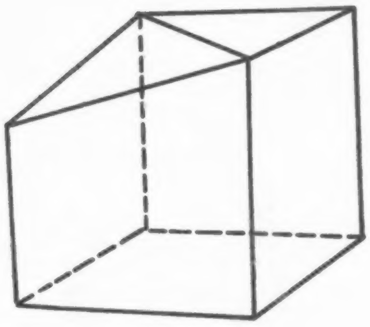
We next turn to the reduction assumption in part (2) of Theorem 2 above.

4. In determining the maximal possible number of facets of a d-polytope P with v vertices, we may assume without loss of generality that P is a simplicial polytope.

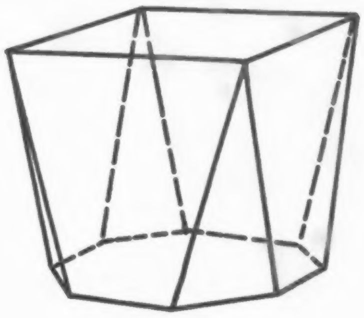
An argument that comes to mind at once to establish this result is of the following type: If there is a facet F which is not a simplex, then F admits a (d-2)-diagonal G. Including in the set of intersecting halfspaces which determine P a halfspace which is close to F^+ but rotated slightly about G we will "slice off" a piece of P. For example, if F is the top face of a cube, then



becomes



If the amount "cut off" is small, so that the number of vertices remains the same, then the number of facets is increased, and repeated such operations gives a simplicial polytope and 4 is true. However, the following example shows that it is not obvious that this sort of thing can be done without an increase in the number of vertices.



Another (and successful) way to approach the proof of 4 is by the use of the following theorem (Eggleston-Grünbaum-Klee [1]).

5. For each d-polytope P, there exists an $\epsilon > 0$ such that if P' is another d-polytope with $\rho(P,P') < \epsilon$, then $f_k(P') \geqq f_k(P)$ for all k. The Hausdorff metric ρ is defined by

$$\rho(P,P') = \inf\{\alpha > 0 \mid P \subset (P'+\alpha B) \text{ and } P' \subset (P+\alpha B)\}$$

where B is the unit ball at the origin.

This theorem gives us the reduction to simplicial polytopes as follows. If P is not simplicial, then there are too many vertices on one facet. It is then possible to move one of the vertices away from P by an amount sufficiently small to assure that all other vertices remain vertices. This "pulling out" operation does not increase the number of vertices. The combinatorial structure might have changed quite a lot, but we are interested in only $f_{d-1}(P')$, and Theorem 5 implies $f_{d-1}(P') \geqq f_{d-1}(P)$ as long as the distinguished vertex moves by less than $\epsilon(P)$. If P' still has non-simplicial facets we continue this "pulling out" process, at each step changing each successive polytope by a very small amount (with respect to the Hausdorff metric), keeping the number of vertices the same, and increasing (non-strictly) the number of facets until we obtain a simplicial polytope. This establishes Theorem 4.

It is well-known that if $\rho(P,P')$ is small, and if Ψ_k denotes the k-dimensional content of the k-faces, then $\Psi_d(P)$ and $\Psi_d(P')$ are close, and $\Psi_{d-1}(P)$ and $\Psi_{d-1}(P')$ are close. Easy examples show that in general the (d-2)-dimensional contents need not behave similarly: in E^3 let P be a cube and P' the similar cube with one of its edges "sliced off." However, it may be proved that $\Psi_k(P)$ is a lower semicontinuous function of

P for all k, and that $\Psi_k(P)$ may be extended to the family of all compact convex sets under preservation of this property (allowing value $+\infty$) (Eggleston-Grünbaum-Klee [1]). This observation leads to a wealth of isoperimetric type problems concerned with d-polytopes such as the following. For a given d-polytope P whose k-faces have k-dimensional content $\Psi_k(P) = 1$, what can be said about $\Psi_i(P)$ for $i \neq k$, $1 \leq i \leq d$? In particular, no one has proved that the convex set P in E^3 with largest volume, for which $\Psi_1(P) = 1$, must be a polytope and not have an infinite number of vertices, nor is the maximal volume known.

We now turn to a generalization of the third step of the proof of Theorem 2. We noted that for simplicial polytopes in E^3, $2f_1 = 3f_2$. The generalization again counts certain objects in two different ways. For $0 \leq i,j \leq d$, and for a d-polytope K, let $g_{ij}(K)$ denote the number of incidences of an i-face of K with a j-face. The object we count in two different ways is

$$(*) \qquad \sum_{i=j}^{d-1} (-1)^i g_{ij}(K) \qquad \text{for } j \text{ fixed.}$$

Recall that when we changed the problem from that of finding a bound on the number of vertices to that of finding a bound on the number of facets of a polytope, we mentioned that it was the natural duality that made these problems equivalent. We again make use of this duality. A k-face (and its faces) is itself a k-polytope which satisfies Euler's formula. In the dual setting the k-face becomes a (d-1-k)-face of the dual polytope and each i-face of the k-face becomes a (d-1-i)-face of the dual which contains the (d-1-k)-face. Let $h_i(F)$ denote the number of i-faces of K incident with the k-face F of K. Because of this duality, these must satisfy the Euler relation

$$\sum_{i=k}^{d-1} (-1)^i h_i(F) = (-1)^{d-1}.$$

Note that this is just Euler's formula when $k = -1$, $F = \emptyset$ and $h_i(F)$ becomes $f_i(K)$. Using this formula we get

$$\begin{aligned}(*) &= \sum_{i=j}^{d-1} (-1)^i \sum h_i(F) \\ &= \sum \sum_{i=j}^{d-1} (-1)^i h_i(F) \\ &= (-1)^{d-1} f_j,\end{aligned}$$

where Σ indicates summation over all j-faces F. On the other hand, counting the same thing, but considering i-faces rather than j-faces we get

$$\begin{aligned}(*) &= \sum_{i=j}^{d-1} (-1)^i \sum h_j(F^{(i)}) \\ &= \sum_{i=j}^{d-1} (-1)^i \binom{i+1}{j+1} f_i ,\end{aligned}$$

where Σ indicates summation over all i-faces $F^{(i)}$. This last equality comes from answering the question "How many j-simplices are determined by an i-face?" Since the reduction to simplicial polytopes assures us that each i-face is an i-simplex, it is clear that $h_j(F^{(i)}) = \binom{i+1}{j+1}$. Both of these formulas are valid for each fixed j where $-1 \leqq j \leqq d-2$. Combining them we get the equations

$$(**) \qquad \sum_{i=j}^{d-1} (-1)^i \binom{i+1}{j+1} f_i = (-1)^{d-1} f_j .$$

It would be nice here if these d equations in the d variables $f_0, f_1, \ldots, f_{d-1}$ would allow us to obtain f_{d-1} algebraically as a function of f_0. However they were obtained under the assumption that K was a simplicial polytope, and it turns out that only $\left[\frac{d+1}{2}\right]$ of these equations are independent. ($[z]$ denotes the greatest integer in z.) We state this result and the corresponding result about all d-polytopes in a more formal form. For each d-polytope K let $f(K)$ denote the vector $f(K) = (f_0(K), f_1(K), \ldots, f_{d-1}(K)) \in E^d$.

6. The dimension of the affine hull of the set $\{f(K) \mid K \text{ is a d-polytope}\}$ is d-1.

7. The dimension of the affine hull of the set $\{f(K) \mid K \text{ is a simplicial d-polytope}\}$ is $[d/2]$.

Note that Euler's formula requires that the dimension of the flat in 6 be at most d-1, since each vector $f(K)$ lies on the hyperplane $H(u,\alpha) \subset E^d$ where $u = (+1,-1,+1,-1,\ldots,(-1)^{d-1}) \in E^d$ and $\alpha = 1 - (-1)^d$. Proofs of these results are omitted.

Even though only about half of the equations (**) are independent, they will still be useful in the following form. It is possible to find independent equations giving each f_i, $i \geqq [d/2]$ in terms of the f_i, $i < [d/2]$. For example in E^4 these equations become

$$\begin{aligned} f_2 &= 2f_1 - 2f_0 \\ f_3 &= f_1 - f_0, \end{aligned}$$

in E^5 the equations become

$$\begin{aligned} f_2 &= 4f_1 - 10f_0 + 20 \\ f_3 &= 5f_1 - 15f_0 + 30 \\ f_4 &= 2f_1 - 6f_0 + 12, \end{aligned}$$

while in E^6 the equations become

$$\begin{aligned} f_3 &= 3f_2 - 5f_1 + 5f_0 \\ f_4 &= 3f_2 - 6f_1 + 6f_0 \\ f_5 &= f_2 - 2f_1 + 2f_0. \end{aligned}$$

These equations were first found in 1905 by Max Dehn [1] for d = 4 and 5 in a complicated way as the by-product of some other results. In 1927

Sommerville [1] worked out similar equations in E^d. These were ignored (even by Sommerville in his 1929 book [2]) until the 1960's. In 1962 Klee [3] independently found these equations and more besides.

While the Dehn-Sommerville equations (**) will prove useful, they still do not allow us to find $\mu(v,d)$ for large values of d. What is now needed is to get some relation between f_0 and f_i for all $0 \leqq i < [d/2]$ so that the Dehn-Sommerville equations may be expressed in terms of f_0 above. As a first (and apparently crude) estimate, note that each i-face has $i+1$ vertices and for two i-faces to be different they must differ in at least one vertex, so $f_i \leqq \binom{f_0}{i+1}$ for $i \leqq d$. In a certain rather surprising sense these are also best possible when $i < [d/2]$, namely, there exist certain polytopes, the cyclic polytopes among others, for which equality holds in these cases. We will use this fact, together with the Dehn-Sommerville equations to evaluate $\mu(v,d)$ in some cases. But first we investigate the cyclic polytopes following Gale [1,2].

Let $t \in E^1$ be a real parameter and define $p(t) \in E^d$ by $p(t) = (t,t^2,\ldots,t^d)$. The set $\{p(t) \mid t \text{ is real}\}$ is called the moment curve in E^d. If $t_1 < t_2 < \cdots < t_v$ are any $d+1$ or more distinct values of t, then the d-polytope $C(v,d) = \text{conv}\{p(t_1),\ldots p(t_v)\} \subset E^d$ is called a cyclic polytope. It may be shown that the combinatorial type of $C(v,d)$ is not dependent upon the particular choices of the t_i.

8. Let $C(v,d)$ be a cyclic polytope with any number $v \geqq d+1$ of vertices. For any number $n \leqq [d/2]$, let $V_n = \{p(t_1),\ldots,p(t_n)\}$ be an arbitrary subset of n vertices of $C(v,d)$. Then V_n determines an $(n-1)$-face of $C(v,d)$. Therefore, $f_i(C(v,d)) = \binom{v}{i+1}$ if $0 \leqq i < [d/2]$.

To show this we give the equation of a hyperplane H and note that each

$p(t_i) \in V_n$ lies on H, while all other vertices $p(t)$ lie in one open half-space determined by H. Consider the polynomial

$$0 \leqq \Pi_{i=1}^{n} (t-t_i)^2 = \beta_0 + \beta_1 t + \cdots + \beta_{2n} t^{2n}$$

where t is the variable and each β_i depends only on the constants $t_1,\ldots,t_n$ which determine the points of V_n. Note that $t_1,\ldots,t_n$ are the only roots of the polynomial and the polynomial is non-negative. Thus if $b = (\beta_1,\beta_2,\ldots,\beta_{2n},0,0,\ldots,0) \in E^d$, we see that $\langle p(t),b\rangle \geqq -\beta_0$ for all t and equality holds if and only if $t = t_i$ for some i. Thus $H(b,-\beta_0)$ is the hyperplane, and the proof is complete.

It is easy to show that the cyclic polytopes are simplicial polytopes. This is useful because it implies that the Dehn-Sommerville equations hold for cyclic polytopes. Furthermore, using just the properties of $C(v,d)$, it is possible to calculate the number f_i of i-dimensional faces of $C(v,d)$. In particular, the general formulas for the (d-1)-faces of cyclic polytopes are

$$f_{d-1}(C(v,d)) = \frac{v}{v-n}\binom{v-n}{n} \qquad \text{if } d = 2n$$

$$f_{d-1}(C(v,d)) = 2\binom{v-n-1}{n} \qquad \text{if } d = 2n+1.$$

In E^4 if we apply $f_i \leqq \binom{v}{2} = \binom{f_0}{2}$ to the Dehn-Sommerville equations we get

$$f_2 \leqq f_0(f_0-3)$$

$$f_3 \leqq \tfrac{1}{2} f_0(f_0-3)$$

and Theorem 8 shows that equality holds for the cyclic polytopes. This gives the complete solution to our problem if $d = 4$, and similar reasoning works when $d = 5$. To summarize: letting $\mu_k(v,d)$ equal the maximum possible number of k-faces in any d-polytope with v vertices, we have

9. $\mu_3(v,4) = \frac{1}{2}v(v-3)$

$\mu_2(v,4) = v(v-3)$

$\mu_1(v,4) = \frac{1}{2}v(v-1)$

Similarly,

10. In E^5, since $f_4 = 2f_1 - 6f_0 + 12$ and $f_1 \leqq \binom{f_0}{2}$ with equality holding for $C(f_0,5)$, it follows that $\mu_4(v,5) = v^2 - 7v + 12$.

Analogous equations may be found for $\mu_k(v,5)$, $1 \leqq k \leqq 3$. Unfortunately, this method already breaks down if $d = 6$. We know that $f_2 \leqq \binom{f_0}{3}$ and $f_1 \leqq \binom{f_0}{2}$ are true (and that equality holds for cyclic polytopes) but these are not of use in the Dehn-Sommerville equation $f_5 = f_2 - 2f_1 + 2f_0$ because the f_i have alternating signs in this equation. For larger values of d even the Dehn-Sommerville equations (**) get out of hand. It is possible to solve for f_{d-1} in general in terms of the f_i for $i < [d/2]$, and note that the coefficients of the f_i have alternating signs. But for values of k strictly between $[d/2]$ and $d-1$, the coefficients of f_i in the representation of f_k are so complicated in general that no one has even proved that they have alternating signs.

For this reason we now concentrate on finding $\mu(v,d) = \mu_{d-1}(v,d)$ for the larger values of d. The appropriate Dehn-Sommerville equation for this case is easier to express in the even and odd case separately: If $d = 2n$

$$f_{2n-1} = \sum_{i=0}^{n-1} (-1)^{n+i+1}\left(\frac{i+1}{n}\right)\binom{2n-2-i}{n-1}f_i$$

and if $d = 2n+1$

$$f_{2n} = 2\sum_{i=-1}^{n-1} (-1)^{n+i+1}\binom{2n-1-i}{n}f_i.$$

We also know that $f_i \leqq \binom{f_0}{i+1}$ if $0 \leqq i < [d/2]$ in simplicial polytopes with equality holding for the cyclic polytopes. It was this latter result that

Klee generalized in 1962, thereby providing the major break-through that lets us determine $\mu(v,d)$ in most cases. Klee's result is (see Klee [2])

11. For every simplicial polytope, f_s is related to f_{s-1} by the formula: $(s+1)f_s \leqq (f_0 - s)f_{s-1}$. Furthermore, this is independent of the dimension of the polytope, and equality holds if and only if the polytope is $(s+1)$-neighborly (i.e., each $(s+1)$ vertices determine an s-face).

This result is proved by the above process of counting things in two different ways, and it may be extended to the following relation between f_s and f_{s-j}

$$\binom{s+1}{j} f_s \leqq \binom{f_0+j-1-s}{j} f_{s-j}$$

for $0 \leqq j \leqq s$. In particular, equality holds in the above formulae for the cyclic polytopes when $0 \leqq s < [d/2]$ (because cyclic polytopes are $[d/2]$-neighborly).

As an example, we will illustrate how this result can be used to obtain an upper bound for f_{d-1} for simplicial polytopes when $d = 8$.

a) $f_7 = f_3 - 3f_2 + 5f_1 - 5f_0$	Dehn-Sommerville equation
b) $4f_3 \leqq (f_0-3)f_2$	Theorem 11 for simplicial polytopes
c) $f_7 \leqq f_3\left(1 - \frac{12}{f_0-3}\right) + 5f_1 - 5f_0$	a) and b)
d) $\leqq \left(1 - \frac{12}{f_0-3}\right)\binom{f_0}{4} + 5\binom{f_0}{2} - 5f_0$ if $f_0 \geqq 15$	$f_i \leqq \binom{f_0}{i+1}$, and assuming that $\left(1 - \frac{12}{f_0-3}\right) \geqq 0$, i.e., $f_0 \geqq 15$.
e) $= f_7(C(f_0,8))$ if $f_0 \geqq 15$	all inequalities are equalities in the case of cyclic polytopes.

It is possible that the term $(1 - 12/(f_0-3))$ in d) might be negative, and thus make the estimate in the step to d) invalid. But this can only happen if $f_0 < 15$. Hence the above proof is valid only if $f_0 \geqq 15$. Similar arguments work in the case of general d provided

$$(***) \qquad f_0 \geqq \begin{cases} n^2 - 1 & \text{where } d = 2n \\ (n+1)^2 - 2 & \text{where } d = 2n+1 \end{cases}$$

For any particular value of d, these results can also be extended to get values of f_k for all intermediate values of k, provided, as in the above case, that f_0 is sufficiently large compared with d.

The above arguments show that for all polytopes P in E^d with sufficiently many vertices, the maximum of $f_{d-1}(P)$ is assumed when P is a cyclic polytope. The question now arises, when do the cyclic polytopes determine the maximum value of f_k (for fixed f_0 and d) over all polytopes? The upper bound conjecture asserts that this is always the case. Stated in other terms, we know that $f_k(C(v,d)) \leqq \mu_k(v,d)$; the upper bound conjecture asserts that equality always holds. The following theorem lists all the cases for which this conjecture has been established (by an elaboration of the above methods, or by other means).

12. The upper bound conjecture $\mu_k(v,d) = f_k(C(v,d))$ is true at least in the following cases:

(1) for every k, $1 \leqq k \leqq d-1$, provided v is sufficiently large

(2) for every k, $1 \leqq k \leqq d-1$, provided $v \leqq d+3$

(3) for $k = d-1$ provided

$$v \geqq n^2 - 2 \qquad \text{if } d = 2n$$

$$v \geqq (n+1)^2 - 3 \qquad \text{if } d = 2n + 1$$

(4) for $k = [d/2]$ provided

$$v \geqq \tfrac{1}{2}(n^2 + 3n - 6) \qquad \text{if } d = 2n$$

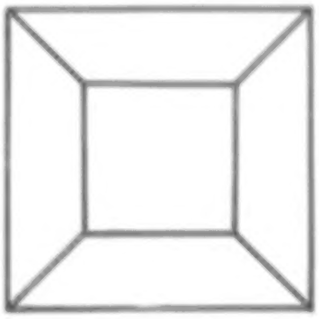
$$v \geqq \tfrac{1}{2}(n^2 + 5n - 4) \qquad \text{if } d = 2n+1$$

(5) for all (allowable) v and k provided $d \leqq 8$.

The values of $\mu_k(v,3)$ and $\mu_k(v,4)$ have been known since at least the first of this century. The cases for $d = 5$ and 6 were established in 1961. The most important parts of Theorem 12 were established in separate papers by Klee [2] and Gale [3] in 1964. The upper bound conjecture itself is due to Motzkin [1] who categorically stated in a 1957 research announcement that it was true. But since no detailed exposition has appeared in the intervening years it seems reasonable to refer to it as a conjecture.

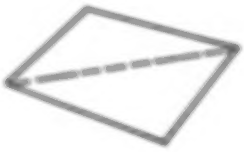
Discussion.

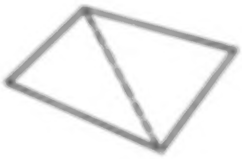
Johnson pointed out that the first argument of Theorem 4 is further complicated by the fact that there exist non-convex simply-connected polytopes in E^3 for which there is no simplicial subdivision by diagonals. He also asked if Theorem 3 could be proved by use of Schlegel diagrams. (Definition: A Schlegel diagram of the d-polytope P is obtained by choosing a point $x \notin P$ sufficiently near the interior of a facet F of P, and then projecting P radially from x onto the facet F. This gives a $(d-1)$-dimensional representation of the faces of P, all of which (except F itself) appear in a cellular division of F. For example: a 2-dimensional Schlegel diagram of the cube in E^3 is shown. (Note that it is what you see if you "peek in a face.") Grünbaum pointed out that the questionable assumption in the



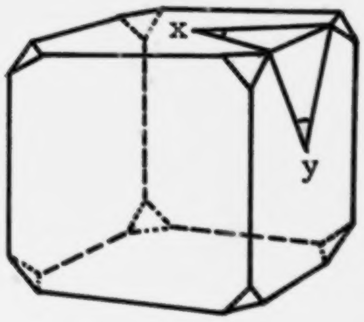
early proofs of Euler's formula (Theorem 3) is still far from obvious in general, even in this reduced 2-dimensional case.

Grünbaum also presented the following related problem posed by J. Steiner more than a century ago, which shows a difficulty in the idea of "slightly moving vertices." Given a d-polytope P, can you find a d-polytope P' which is combinatorially equivalent to P (i.e., $\mathfrak{F}(P)$ and $\mathfrak{F}(P')$ are isomorphic lattices) and which has all its vertices on a sphere? In his 1900 book, Brückner stated the answer was obviously "yes" if the polytope was simplicial. His false argument said: project P to a containing sphere, and take P' to be the convex hull of the projections of the vertices. The main trouble occurs in that you may wish that four particular points on the sphere determine the triangles shown above, while in fact they determine the triangles shown below. This projection provides a homeomorphism but the question is whether the cells have affine representation, and if this is made to be so, is the resulting figure still convex? In 1928 Steinitz [2] showed not only that Brückner's proof is false, but even proved there are lots of polytopes (simplicial ones among them) for which there does not exist any representation with vertices on a sphere. Steinitz's proof is more readily explained in the dual formulation involving 3-polytopes with an inscribed sphere. There is a wide class of 3-polytopes which do not have any realization with an inscribed sphere meeting each face. Start with any polytope which has more vertices than faces, for example, the

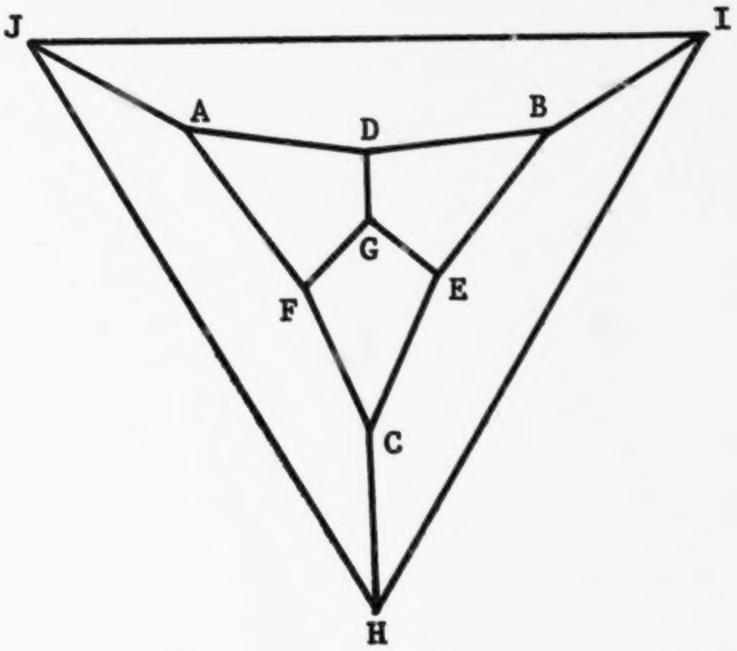




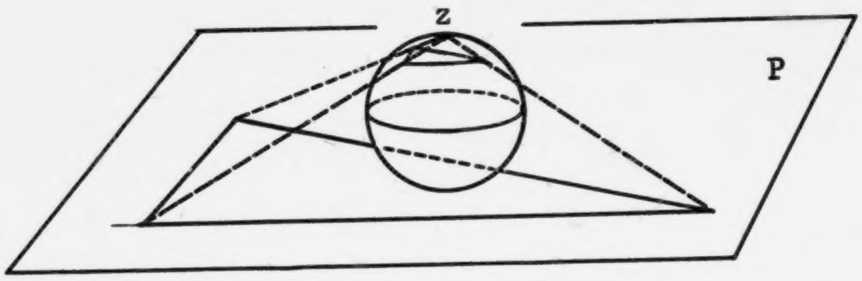
cube. Slice off each vertex to form v new faces. If only a little is sliced off then these new faces will be pairwise disjoint. Assume such a polytope P has an inscribed sphere. Each edge subtends the same angle from the two points of tangency in the adjoining faces. But since there are more "newer" faces than "old" the sum of the subtended angles in the new faces must be more than the same sum for the old faces. But this implies that two "new" faces must have an edge in common, a contradiction. The dual of any such example becomes an example where the vertices cannot be inscribed in a sphere.



A different approach establishes the existence of a family of simple 3-polytopes which have no combinatorially equivalent representative inscribed in a sphere. To explain this approach, consider K, a cube in E^3 with one vertex sliced off. If we "peek in" the triangular face, the resulting Schlegel diagram for this polytope is clearly seen to be



Assume that this sliced cube had each vertex on a sphere S. Also assume that the projection used to obtain the figure above was of the following special sort. Take a point z on S above the triangle, and take the usual stereographic



projection onto the opposite supporting plane P. This stereographic projection sends circles on S (not containing z) onto circles in P. The quadrilateral ADGF $\subset$ P is thus inscribed in a circle (the image of the circle which is the intersection of S with the corresponding facet of P). Similarly the quadrilaterals BDGE and CEGF are inscribed in circles, as are the pentagons ADBIJ, BECHI, AFCHJ. Using the fact that opposite angles of any quadrilateral inscribed in a circle add up to 180°, it follows that D, E, F lie on the sides of the triangle ABC. Thus, for example, A, D, B are collinear, a contradiction to their being on a (proper) circle. This proves that no polytope with the combinatorial type of K can have its vertices on a sphere.

Yale asked what the motivation was for using the word "cyclic" in cyclic polytopes. The moment curve is not the only curve which can be used to generate neighborly polytopes. Another curve with this property and described in terms of the real parameter t is the set of all points in E^{2n} of the form (sin t, cos t, sin 2t, cos 2t,..., sin nt, cos nt). This curve was used sixty years ago by Carathéodory in his studies of analytic functions. Notice that it is a bounded, periodic curve, i.e., it "cycles" in E^{2n}. Carathéodory's work with cyclic polytopes was forgotten until the 1950's, when Gale [1, 2] first independently described neighborly polytopes in a complicated way, rediscovered Carathéodory's work, and then went beyond Carathéodory to get simple proofs to further results on cyclic and neighborly polytopes. The names are due to Gale.

Coxeter suggested that the mathematical community should read $\binom{a}{b}$ as "a choose b" rather than "a over b."

Theorem 11 of Klee is valid for all simplicial complexes. It is still an open problem as to whether this can be extended to non-simplicial polytopes.

11. Related results and problems.

In the last section we worked directly toward an answer to the problem "What is $\mu_{d-1}(v,d)$?" In answering that question we presented many results which lead to interesting generalizations or problems on their own. Several of these are discussed in the present section.

We first consider various generalizations of Euler's formula. There are many similar equations associated with other characteristics of polytopes. The first example we shall give is a generalization of a fact known to Euclid: the sum of the interior angles of an n-gon in E^2 is $(n-2)\pi$, or $(n-2)/2$ full angles. Let P be a d-polytope, F any k-face of P, $0 \leqq k \leqq d$, and denote by $c(F,P)$ the convex cone spanned by P from the vertex at the centroid of F. Let $\varphi(F)$ be the fraction of E^d taken up by the cone $c(F,P)$. (Intersect $c(F,P)$ with the unit sphere whose center is at the vertex of $c(F,P)$ to make this definition precise.) In the planar case where P is a convex polygon,

$$\varphi(F) = \begin{cases} 1 & \text{if } F = P \\ \frac{1}{2} & \text{if } F \text{ is an edge} \\ (\text{the interior angle at } F)/2\pi & \text{if } F \text{ is a vertex.} \end{cases}$$

For each $k = 0,1,\ldots,d$, define the k^{th} angle sum $\alpha_k(P)$ by $\alpha_k(P) = \Sigma\, \varphi\,(F)$ where the sum is taken over all k-faces F of P. Thus the result known to Euclid could be rephrased; for all 2-polytopes P, $\alpha_0(P) = (f_0-2)/2 = (f_1-2)/2 = \alpha_1(P) - 1$. This result has the following generalization.

1. For every d-polytope P, $\Sigma_{i=0}^{d-1} (-1)^i \alpha_i(P) = (-1)^{d-1}$.

The special case $d = 3$ was discovered by Gram in 1874. It is well-known that in E^3, if all faces of P are of the same type, then they must be either triangles or quadrilaterals or pentagons. No generalization of this fact for higher dimensions was known until recently when Perles and Shephard [1] obtained results of this type using these angle sums $\alpha_k(P)$. Even if we know that all facets of a 4-polytope are 3-polytopes of a particular combinatorial type, there is no bound on the number of vertices of these 3-polytopes in different examples. Perles and Shephard showed the possibility of using certain classes of polytopes as facets and the impossibility of so using certain other classes.

It is interesting to note that the equation of Theorem 1 is unique in the following sense.

2. If $\Sigma_{i=0}^{d-1} (-1)^i \beta_i \alpha_i(P) = \beta_d(-1)^{d-1}$ holds for all d-polytopes P, then $\beta_0 = \beta_1 = \cdots = \beta_d$.

We now turn to an example of a Euler-type equation involving d-polytopes which is a vector equation rather than a scalar equation. For each vertex $x \in P$, consider the cone generated by the outward normals to all hyperplanes which support P at x. As in the last example, consider the content $\psi(x)$ of this exterior cone. For obvious geometrical reasons $\psi(x)$ is called the exterior angle of P at x. We form the vector sum of these vertex points using the content of the cones as weights, and define the resulting vector $S(P)$ to be the Steiner point of the polytope P. That is,

$$S(P) = \Sigma\, \psi(x)\, x \quad \text{(summed over all 0-faces } x\text{)}.$$

The most useful properties of $S(P)$ are given by

3. (a) $S(P)$ depends upon neither the dimension of the space which contains P nor the location of the origin in that space. (b) For all real α, β and all polytopes P, Q,

$$S(\alpha P + \beta Q) = \alpha S(P) + \beta S(Q)$$

where addition on the left is vector addition.

Now each i-face $0 \leqq i \leqq d-1$, is also itself a polytope and therefore admits its own Steiner point $S(F)$. For each i, $0 \leqq i \leqq d-1$, let us define a vector $\mu_i(P)$ by

$$\mu_i(P) = \Sigma\, S(F) \quad \text{(summed over all i-faces } F \text{ of } P).$$

Note that $\mu_i(P)$ definitely depends upon the location of the origin and in general will not be in the polyhedron P. Yet taken together they satisfy the Euler-type vector equation

$$4. \quad \Sigma_{i=0}^{d-1} (-1)^i \mu_i = (1 + (-1)^{d-1}) S(P).$$

It has recently been shown by Sallee [1] that the Steiner points also have a nice valuation property, namely, when P_i are polytopes and $P_1 \cup P_2$ is convex $S(P_1) + S(P_2) = S(P_1 + P_2) = S(P_1 \cup P_2) + S(P_1 \cap P_2)$.

The points $S(P)$ have been called Steiner points because in 1840 J. Steiner proposed studying the center of gravity of a mass distribution on a curve in E^2, where the mass at a point on the curve is proportional to the curvature at the point. The Steiner point as defined here for a polytope is the discrete analog of the center of gravity of such a mass distribution.

It might be noted that if the support function of a convex set is developed in a Fourier series, then a translation of the origin changes only the first-order coefficients of the Fourier series. If the origin is translated to the Steiner point then these first order coefficients will be zero. Similarly in higher dimensions, if the support function is developed in spherical harmonics of the appropriate dimension, the origin is at the Steiner point if and only if the first order terms are zero. Since the supporting functions add when convex sets are added, it is thus reasonable to expect that $S(P + Q) = S(P) + S(Q)$.

* * * *

We next turn to a generalization of Klee's inequality (Theorem 10.11) which is valid for all simplicial complexes rather than just those simplicial complexes realized by simplicial polytopes. If a given simplicial complex has f_k k-faces, it is natural to ask what is the best possible lower [respectively, upper] bound on the number f_i of i-faces of the complex, when $i \leqq k$ [respectively, $i \geqq k$]. This problem has been solved by J. B. Kruskal [1] using a rather complicated proof. We will illustrate Kruskal's result by an example. Suppose it is known that in a given simplicial complex $f_4 = 30$. We write f_4 in a certain canonical form:

$$f_4 = \binom{a}{5} + \binom{b}{4} + \binom{c}{3} + \binom{d}{2} + \binom{e}{1}.$$

In general the canonical form of f_k starts with $\binom{a}{k+1} + \binom{b}{k} + \cdots$. We choose the constants $a,b,c,\ldots$ as follows. Choose a as large as possible so that $f_4 \geqq \binom{a}{5}$. Having chosen a ($a = 7$ in this example) choose b as large as possible so that $f_4 \geqq \binom{a}{5} + \binom{b}{4}$. Continue this process until equality is reached at some stage. Thus $f_4 = 30 = \binom{7}{5} + \binom{5}{4} + \binom{4}{3}$ is a canonical representation. Then for i, the best possible bounds on f_i are given by $f_i = \binom{a}{i+1} + \binom{b}{i} + \binom{c}{i-1} + \cdots$ where this representation continues as long as the representation of f_4. (Terms like $\binom{n}{n+j}$ are zero terms, $j \geqq 1$.) Thus in our example,

$$f_3 = \binom{7}{4} + \binom{5}{3} + \binom{4}{2} = 51$$

and

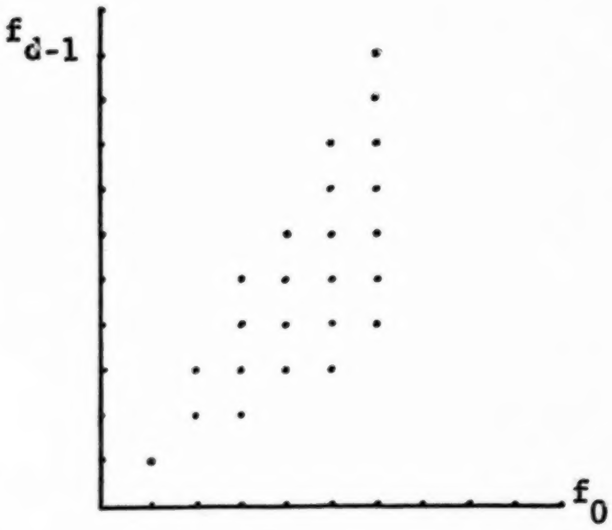
$$f_6 = \binom{7}{7} + \binom{5}{6} + \binom{4}{5} = \binom{7}{7} = 1.$$

It must be emphasized that these are the best possible bounds for the class of all simplicial complexes. Even though they are bounds for the f_i in terms of f_k for polytopes, they are generally not the best bounds. Nevertheless, Kruskal's result has proved its usefulness as a tool in the theory of polytopes by helping to establish several cases of the upper bound conjecture. In particular it allows us to establish the upper bound conjecture in the case $d = 7$, and the case $d = 8$ follows from the case $d = 7$ by a rather simple argument.

It would be interesting to know how Kruskal's results might be extended when extra geometric conditions are added. We know that we can embed topologically, or even rectilinearly, each simplicial k-complex in E^{2k+1}. But suppose that a simplicial k-complex can be embedded in E^{k+1}. This adds restrictions which conceivably can be used to obtain better bounds on the f_i.

* * * *

A problem similar to the one considered in Section 10 is, Given a d-polytope with v vertices, how small may f_k be? The answer is easy if $d = 3$, and is known for most cases if $k = d-1$, but almost nothing is known for intermediate values of k. We first show that as much is known about the lower bounds on f_{d-1} as the upper bounds. For a fixed dimension d, plot $f_0 > d$ against f_{d-1}, placing a dot on the graph whenever there is a d-polytope having this number of vertices and facets. It is clear from the duality which takes f_0 into f_{d-1} that this graph must be symmetric about the diagonal. Thus the lower bounds on f_{d-1} can be found from knowing the upper bounds on f_{d-1}. This technique does not work for f_{d-2}. In the dual setting we are then asking for upper bounds on f_{d-1} in



terms of f_1, the number of edges. Thus these intermediate cases for $k < d-1$ do not reduce to the results of Section 10.

For the upper bounds on f_{d-1} we saw that it was sufficient to look at only simplicial polytopes. It would be nice to get a similarly restricted class of polytopes in which the lower bounds are always assumed. Unfortunately the simplicial polytopes do not have this property. In fact no reasonable conjecture has even been made as to a class of d-polytopes which minimize $f_k(P)$. Nevertheless, it is still interesting to restrict our attention to the simplicial polytopes and ask for the lower bounds on f_k there. A certain adding process which builds up simplicial polytopes in E^3 has lead to specific conjectures in E^d as to what the lower bound for f_{d-1} should be for simplicial polytopes. In particular, suppose that a particular simplicial polytope in E^d with a given number of vertices minimizes f_{d-1}. If we add an extra vertex we must add at least d edges, and similarly at least $(d-1)$ facets. This process has lead to the lower bound conjecture: If $\nu_k(v,d)$ is the best lower bound on f_k for all simplicial d-polytopes with v vertices, then

$$\nu_k(v,d) = \binom{d}{k}v - \binom{d+1}{k+1}k \qquad \text{for } 1 \leqq k \leqq d-2$$
$$\nu_{d-1}(v,d) = (d-1)v - (d+1)(d-2).$$

This is easily checked in E^3, but in E^4 it is not at all clear that the process of adding an extra vertex gives analogous results. The lower bound conjecture has, in fact, only been proved for those cases where an actual enumeration was made for all polytopes with this number of vertices. Perles has a few days ago reported that the lower bound conjecture is true whenever there are at most $d+9$ vertices, thus in particular when $v_0 \leqq 13$. The methods used do not seem to allow for significant improvement.

* * * *

One immediate application of results of the type of Section 10 is to the coloring problems. The famous 4-color problem has been generalized in different ways to higher dimensions, and also to surfaces other than the sphere in E^3. In the higher dimensions most formulations of the problem do not admit upper bounds in the sense that no finite number of colors is sufficient to color all examples. The coloring problem generally turns out to be solvable on surfaces in E^3 other than the sphere. A minor reduction of the 4-color problem is easily obtained, namely it is equivalent to the coloring of the 2-faces of all simple 3-polytopes (i.e., polytopes with each vertex trivalent).

The duals of the cyclic polytopes in E^4 show that there are 4-polytopes with as many 3-faces as desired, each two having a common 2-face. The Schlegel diagram in E^3 of each such 4-polytope gives an example of as many 3-polytopes as desired in E^3, properly meeting at their boundaries, and each two of them will have a 2-face in common. This was independently discovered by Eggleston, Besicovitch, and Rado about fifteen years ago.

There is, however, at least one meaningful variant of the 4-color problem for d-polytopes which has a non-trivial solution, namely:

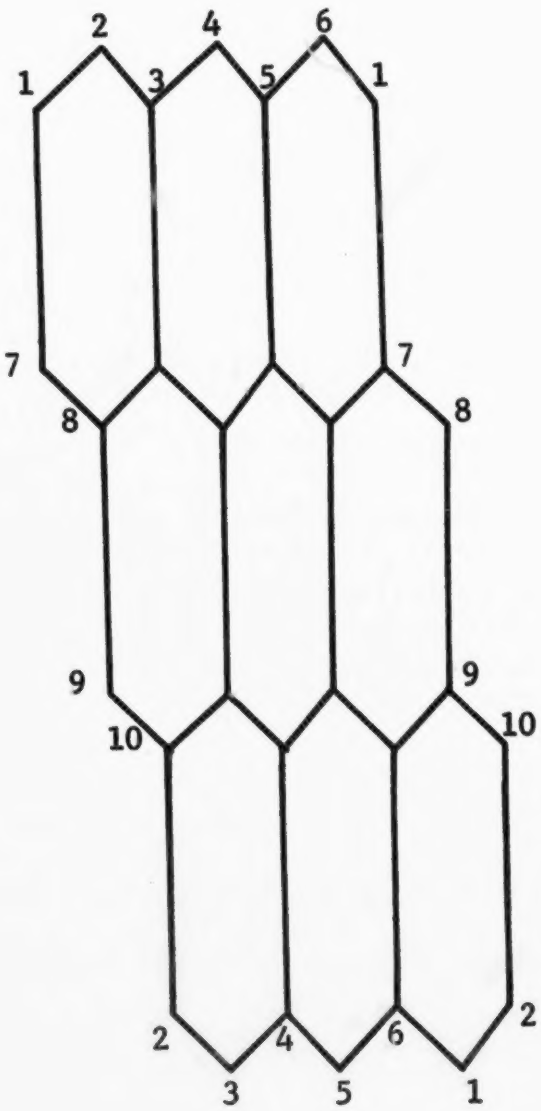
5. The 2-faces of any simple d-polytopes $(d \geqq 4)$ may be colored using at most $6d - 12$ colors. (The coloring is supposed to assign different colors to pairs of 2-faces which have a common edge.)

An open problem in this area that appears to be very difficult is the following. It is well-known that maps in 2-manifolds in E^3 may need a large number of colors if the genus is sufficiently large. If we consider manifolds that are not just topological, but also locally-affine in the sense that each "country" is a planar convex polygon, then there might be an upper limit on the number of colors necessary, independent of the genus of the manifold. A very shakey guess would be that this universal upper limit on the number of necessary colors is 6.

Discussion.

Since many of the results of the last two sections concern cell complexes, there was a discussion on how this is related to algebraic topology. Grünbaum pointed out that usually a topologist is not interested in any particular cell decomposition, but rather quickly passes to subdivisions and uses them as a tool. On the other hand a graph theorist will tend to consider one graph or complex and analyze its particular structure. Perhaps something could be done between these two positions, but many such problems are quite hard. For example, no reasonable conditions on a 2-complex are known which guarantee that it can be embedded in a space of dimension less than 5. Each 2-complex may be embedded in E^5, and if it is also a manifold then it may be embedded in E^4. But it would be nice to know, for example, which 2-complexes may be embedded in E^3, both topologically and piece-wise linearly. Note that even though a refinement of a 2-complex may be geometrically as well as topologically embedded in E^3, this does not imply that the original 2-complex is so embedable. This is in contrast to the one-dimensional case, where every 1-complex may be embedded in E^3, and if it goes into E^2 as well, then it may be embedded geometrically, that is with straight line segments.

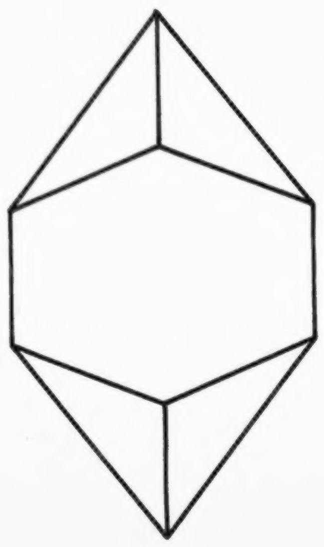
As an example, consider a 2-complex which is the union of 9 hexagons as shown below. This manifold is topologically a torus, with the similarly labeled vertices identified in the obvious way. It is easy to show that if this can be topologically and geometrically embedded (with "flat" hexagons) in any higher dimensional space, then it can be embedded in E^3. Assuming it is embedded in E^3, let us count the sum of the interior face angles of the hexagons in two different ways. The sum of the face angles in each hexagon is 4π. Thus the sum of all interior face angles is $(4\pi)9 = 36\pi$. On the other hand, $6 \cdot$ (# of hexagons) = 3(# of vertices), by counting the number of edges in two different ways. Also, the sum of the interior face angles around a vertex is at most 2π and equals 2π only if the 3 hexagons at that vertex lie in a plane. Combining these results, we can conclude that all the hexagons must be in one plane, a contradiction.



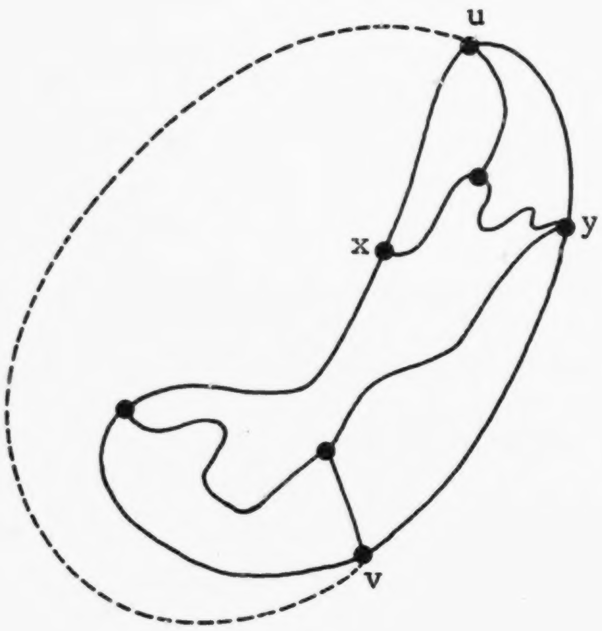
12. Polytopes in E^3.

Much more is known about polytopes in E^3 than about higher dimensional polytopes, in part since it is actually possible to construct models of them. It is much easier to guess what the solutions of a problem should be when there is an actual model to look at. Further, 3-polytopes actually do have a simpler structure than 4-polytopes, just as 1-manifolds are simpler than higher dimensional manifolds.

Part of our familarity with 3-polytopes is due to the possibility of representing them in the plane. The results and methods of graph theory may be applied to the 1-skeleton (set of vertices and edges) of the Schlegel diagram of a 3-polytope. It is clearly seen that the Schlegel diagram of a 3-polytope is planar. The converse question was open for a long time, namely; which configurations in E^2 of convex polygons meeting properly on their edges will be Schlegel diagrams of 3-polytopes? We want each such configuration to have at least four vertices, and each vertex must be at least 3-valent. Further properties are necessary, since the graph in this figure can obviously not represent all but one of the faces of a 3-polytope.



In this section a graph means a finite set of nodes (vertices) and a set of edges which connect certain pairs of these nodes. A graph is n-connected if for each pair x,y of vertices there are at least n disjoint paths along the edges which connect x to y. (Two such paths are disjoint if they have only x and y in common.) Equivalently, a graph with at least n + 1 nodes is n-connected if any n-1 or fewer nodes and their adjacent edges may be removed, and the graph remains connected. For example, the graph with solid lines is not 3-connected because the removal of x and y and their adjacent edges leaves u and v in separate components. Equivalently, each path from u to v goes through either x or y, and thus there are not three distinct such paths. However, if the dotted line is added as an edge, then the graph becomes 3-connected. We say that a graph is 3-realizable if it is isomorphic to the graph of some 3-polytope.



Another well-known property of the graph of every 3-polytope is that each such graph is 3-connected. About fifty years ago E. Steinitz gave several proofs of the converse statement which is one of the most important and deepest results concerning 3-polytopes. Unfortunately his work was forgotten until comparatively recently.

1. (E. Steinitz [1]). Every planar, 3-connected graph is 3-realizable.

An immediate corollary is

2. Each planar 3-connected graph may be "stretched" into a shape where all its edges are straight and all bounded regions in the plane which it determines are convex polygons. (We interpret our definition of graph to exclude two distinct edges connecting a pair of vertices, and the ends of each edge must be distinct vertices.)

The proof of Theorem 1 is very difficult. We will just present its main ideas and point out the difficulties.

For a graph to be 3-connected, it clearly must have at least six edges, and it will have exactly six edges only when it is the complete graph on four nodes. Since we observe that this graph is 3-realizable (by the tetrahedron), we have started our induction proof on the number of edges of the graph. Hence, assume that we are given a graph with at least seven edges. To sketch how the proof will go, we first establish that the graph has 3-valent faces or vertices. Secondly, we will list two ways to transform a graph G into a graph G' such that if G' is 3-realizable, then a 3-polytope P may be constructed which realizes the graph G. These transformations will be called elementary transformations. Thirdly, we will show that if a 3-connected graph G has a 3-valent vertex on a triangular face, then there exists an elementary transformation of G to a 3-connected graph G' with fewer edges than G. The induction hypothesis implies G' is 3-realizable and hence G is 3-realizable. Fourthly, if G does not have a 3-valent vertex on a triangular face, then a finite sequence of elementary transformations of G yields such a graph, in which case the above argument applies, and the proof is complete.

(1) Let v, e, p denote respectively the number of vertices, edges and facets of a 3-polytope P. Let v_k, p_k denote the number of vertices or facets, respectively with exactly k incident edges. Thus $v = \Sigma_{k\geqq 3} v_k$ and $p = \Sigma_{k\geqq 3} p_k$, so Euler's formula may be given the form

$$4\Sigma_{k\geqq 3} v_k + 4\Sigma_{k\geqq 3} p_k - 8 = 4e.$$

It is also clear that

$$\Sigma_{k\geqq 3} kv_k = 2e = \Sigma_{k\geqq 3} kp_k,$$

or

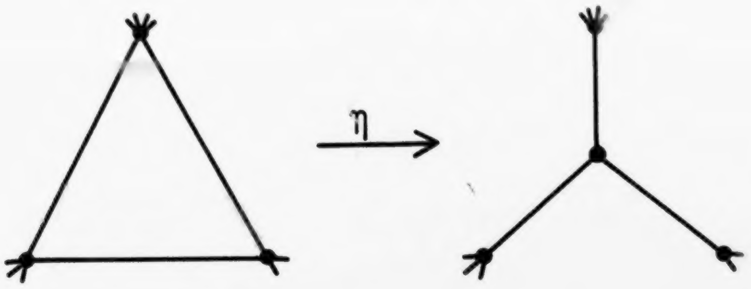
$$4e = \Sigma_{k\geqq 3}\, kv_k + \Sigma_{k\geqq 3}\, kp_k.$$

Combining these two equations we get

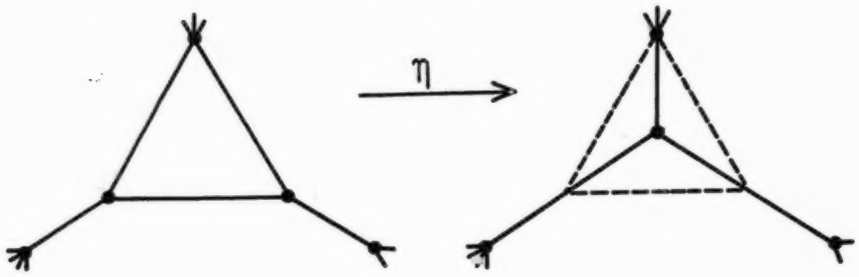
$$v_3 + p_3 = 8 + \Sigma_{k\geqq 5}\, (k-4)(v_k + p_k) \geqq 8.$$

Thus we conclude that each 3-polytope has at least eight 3-valent elements (i.e., 3-valent vertices or triangular faces). Since this argument applies to graphs on the 2-sphere as well, it also applies to planar 3-connected graphs.

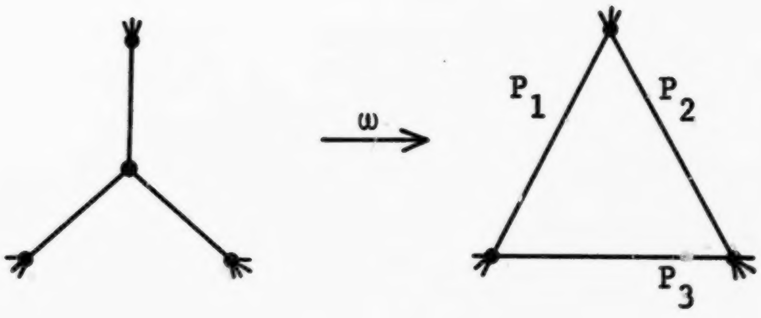
(2) If a 3-connected graph G has a triangular face, as shown on the left below, then we define an elementary transformation η on G to obtain a graph $G' = \eta G$ by removing the three edges of the triangular face and adding one new vertex which is then connected to the three vertices which formed the triangular face; if a vertex of G' is 2-valent, we replace it and the two edges incident to it by a single edge. A typical η-transformation is shown below.

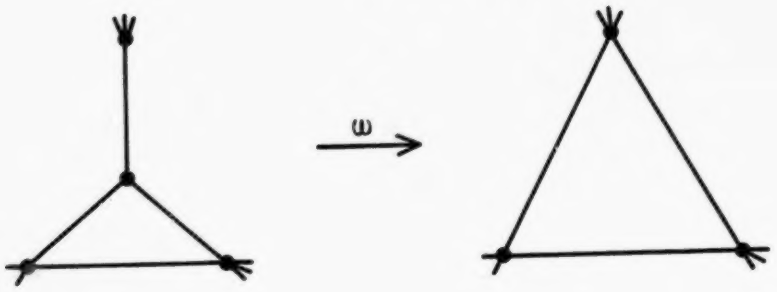


The graph G' is easily seen to be 3-connected. If G' is 3-realizable by a 3-polytope P, it is clear that the "new" vertex may be "sliced off" by a plane through three points corresponding to the three old vertices, thus obtaining a realization of G. Note that if each vertex of the triangular face is at least 4-valent, as shown above, then the number of edges remains the same. However, if one or more of the vertices is 3-valent, as in the example shown below, then the number of edges may be reduced by at least one.



We also define a second type of elementary transformation. A trivalent node and the three edges incident to it are removed from G, and the nodes connected to it are pairwise connected by "new" edges (unless some of them are already edges in G, in which case there is no need for the "new" edge). Examples of such a transformation ω to get a new graph G' are shown below.





If P is a 3-polytope which realizes G' then the faces $P_1P_2P_3$ are "extended" to get a realization of G. As the second example shows, a new face might also need to be formed. It can be shown that it is always possible to obtain a realization of G from any realization of G'. In each case, an elementary transformation does not increase the number of edges.

(3) If a triangular face has at least one 3-valent vertex, then the transformation η reduces the number of edges. In this case the induction completes the proof.

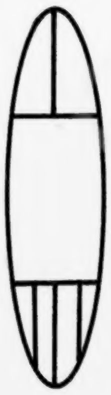
(4) The proof of the fact that some finite sequence of elementary transformations gets us to case (3) is technical and is omitted. It uses special properties of 4-valent, planar graphs. This completes our sketch of the proof.

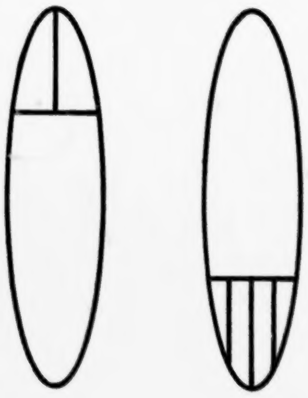
Note that Theorem 1 characterizes which graphs are isomorphic to the graphs of 3-polytopes. It is also possible to characterize in graph theoretic terms graphs which are isomorphic to graphs of centrally-symmetric 3-polytopes, or to graphs of 3-polytopes with a plane of symmetry. We give one other corollary.

3. Every 3-polytope may be obtained from a tetrahedron by a finite sequence of applications of two particular types of operations. (Each operation admits several different cases, as we saw above.)

The next natural question concerns generalizations of Steinitz's theorem to higher dimensions. There are not even any conjectures here about what the conditions in such a theorem should be. For example, suppose a 3-polytope is partitioned into the union of disjoint 3-dimensional convex polytopes which meet properly along their boundaries, and which have appropriate connectedness properties. It is reasonable to ask, does there exist a 4-polytope for which this is its 3-dimensional Schlegel diagram? Such configurations must have certain obvious properties, but there exist examples which satisfy all such known properties, but which are not Schlegel diagrams of any 4-polytope. The smallest simplicial example of this consists of just eight vertices; i.e., a tetrahedron with only four additional interior vertices which define a partition into 19 other simplices.

The 4-color problem concerns the coloring of connected planar regions. If the graph associated with the regions is 3-connected, then Steinitz's theorem implies that it is the graph of a 3-polytope, and coloring the planar regions would be equivalent to coloring the facets of the polytope. (A coloring assumes that regions or faces with a common edge will have different colors, and asks for the smallest number of necessary colors.) If the planar graph is not 3-connected, then easy arguments show that there are two subgraphs whose coloring would imply the coloring of the whole graph. For example, to color the graph at the left, it would suffice to color the two graphs at the right.





The corollary below therefore follows from Steinitz's theorem, although other easier proofs exist.

4. The 4-coloring problem in the plane is equivalent to the corresponding 4-color problem for the faces of 3-polytopes.

* * * *

For about 100 years, people have been trying to count the number of different combinatorial types of 3-polytopes with v vertices or p faces. The problem is hard, and most early papers were attempts to draw pictures, count them, and hope that none were missed. The problem has been significantly advanced recently, and it seems reasonable to believe that within a few years it will be solved, at least for the simple polytopes. ($P \subset E^3$ is simple if each vertex is 3-valent.)

We now consider a simpler problem which was solved by Eberhard [1] in 1891. As above, let $p_k(P)$ be the number of k-gons in a 3-polytope P, and let p be the total number of facets (2-faces). We say that a sequence $(p_3, p_4, \ldots)$ of non-negative integers is 3-realizable provided there is a simple 3-polytope P such that $p_k = p_k(P)$. Since for simple 3-polytopes $v = v_3$, we have

$$3v = 2e = \Sigma_{k \geqq 3}\, kp_k .$$

Together with Euler's formula $v + p = e + 2$ and

$$p = \Sigma_{k \geqq 3}\, p_k$$

we get the following theorem.

5. A necessary condition for a sequence $(p_3, \ldots)$ to be 3-realizable is

$$3p_3 + 2p_4 + p_5 = 12 + \Sigma_{k \geqq 7}\, (k-6)p_k . \qquad (*)$$

It is interesting that (*) implies that there must be at least four faces with five or fewer edges, but it says nothing at all about the number of faces with six edges. Examples show that two sequences may differ only in p_6 and yet one is 3-realizable while the other is not. This leads to the following question. Given a sequence $(p_3, p_4, p_5, p_7, \ldots)$ which satisfies (*), does there exist a value of p_6 which makes $(p_3, p_4, p_5, p_6, p_7, \ldots)$ 3-realizable? Eberhard's theorem says "yes."

6. If $(p_3, p_4, p_5, p_7, \ldots)$ satisfies (*), then there is some p_6 which makes the sequence 3-realizable.

The proof of Eberhard was quite hard, but by using Steinitz's theorem the proof can be simplified. Rather than prove this theorem we will prove another related theorem which is easier, and yet which shows the main idea of the proof. If we work with 4-valent polytopes rather than simple polytopes we must use $4v = 2e$ rather than $3v = 2e$ in the above equations. Similar reasoning then yields

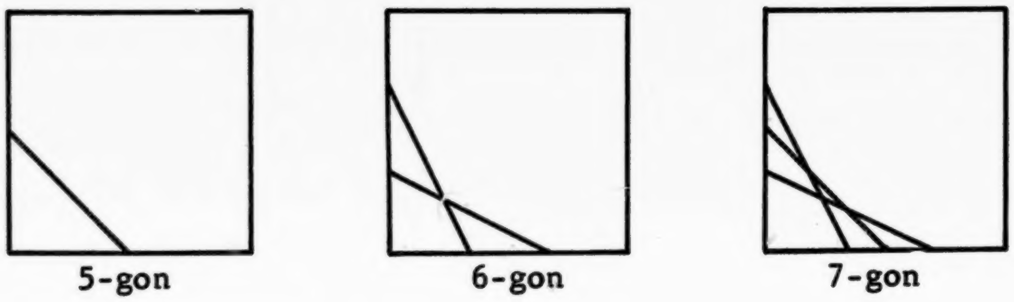
7. If each vertex of P is 4-valent, then

$$(**) \qquad p_3 = 8 + \Sigma_{k \geq 5} (k-4) p_k .$$

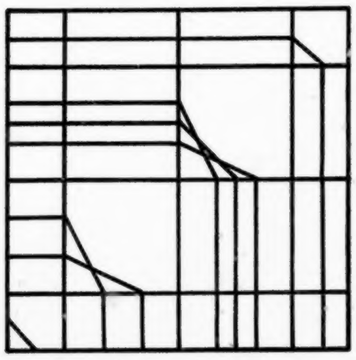
In an analogous way, this leads us to

8. (Grünbaum) If $(p_3, p_5, p_6, \ldots)$ satisfies $(**)$, then there is some p_4 and some 4-valent polytope P such that P has $p_k = p_k(P)$ for all k.

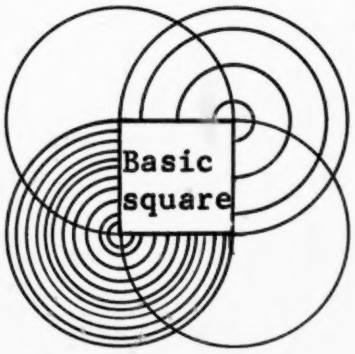
The proof is accomplished by actually constructing a 3-connected, planar graph, with each vertex of valence 4, and with exactly p_k faces with k edges. Then Steinitz's theorem supplies the desired 3-polytope P. Thus we need only construct the graph. Each k-gon that we need, $k \geqq 5$, is formed in the following way from the inside of a square.



It is clear that each such square produces one k-gon, $(k-4)$ triangles, and a number of quadrilaterals. Thus if we use these squares as building blocks of our graph, we will preserve the ratios of p_3 and p_k's in $(**)$. These building blocks are placed together as shown on the left below. They are used as the "diagonal" of a large basic square, and corresponding edge



Basic square



points of the building blocks are connected to the edge of the basic square as shown. We now have a graph which is 4-valent at each vertex except on the boundary of the basic square, and which contains exactly p_k k-gonal regions. To take care of the edge vertices, we close up the basic square with circular arcs, as shown on the right above. This makes each vertex 4-valent, adds lots of quadrilaterals and exactly eight triangular regions as shown. Thus the formula (**) is satisfied by the p_k's of this graph and the proof of the theorem is complete.

A similar type of construction is applicable to the proof of Eberhard's theorem; the details are much more involved, mainly because of the presence of three kinds of "small faces." No result analogous to Eberhard's theorem is known concerning 5-valent polytopes; the equation corresponding to (*) or (**) does not have any p_k missing.

It would be interesting to know what values of p_6 can appear in Eberhard's theorem, and how small the value of p_6 may be. It was earlier conjectured that perhaps $p_6 \leqq n$ where p_n is the last non-zero member of the given sequence. A recent result of Barnette has disproved this conjecture. It has not disproved a conjecture that there is a constant c such that $p_6 \leqq c \sum_{k \neq 6} p_k$. Examples show that c must be bigger than one in this conjecture; probably the best constant is $c = 3$. The 1966 result of Barnette referred to is:

11. If P is a simple polytope, then

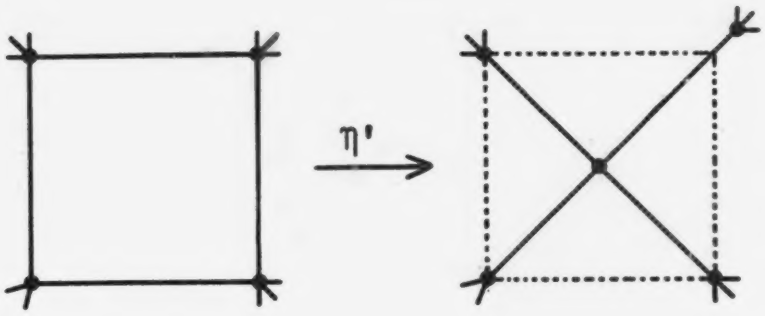
$$2p_3 + 2p_4 + 2p_5 + 2p_6 + p_7 \geqq 8 + \sum_{k \geq 9} (k-8) p_k .$$

In particular, this equation may be used to obtain a lower bound on the size of p_6. There is no similar result known for 4-valent polytopes.

Discussion.

A discussion of the proof of Steinitz's theorem brought out the following facts:

(a) We need to apply the elementary transformation η to a triangular face and not a face with more sides. If η' were the transformation shown below, for example, and if P were a 3-polytope realizing the graph G'



(shown in part on the right), then there is no way to assure that the new vertex may be "sliced off" with a plane through the four old vertices, since they may fail to be coplanar.

(b) In the fourth part of the proof of Steinitz's theorem, we cannot take a particular trivalent vertex and triangular face and guarantee that they can be brought together by a finite sequence of elementary transformations. We only know that such a sequence suffices to bring some pair of 3-valent elements together. The description of this process does, however, give an algorithm by which this can actually be accomplished in a finite number of steps; it is not just an existence statement.

(c) Corollary 2 of Steinitz's theorem may be proved directly with less work than Steinitz's theorem, but it is not a simple fact, and false proofs have been published for it even recently. Direct proofs use only the graph theoretic properties; it does not help to keep track of which bounded regions are convex as the induction proceeds. Yale pointed out that this was remarkable in itself, as it supplies the first known admission by Grünbaum that convexity was not a useful property.

In the discussion it was pointed out that bounds on p_6 in Eberhard's theorem are known for certain special cases. For example, if $p_3 = 4$, then p_6 must be zero or some even number greater than or equal to 4. If $p_4 = 6$, then p_6 must be zero or at least 2. Some similar conditions imply that p_6 may only assume odd values. It was also remarked that Eberhard was blind since his youth.

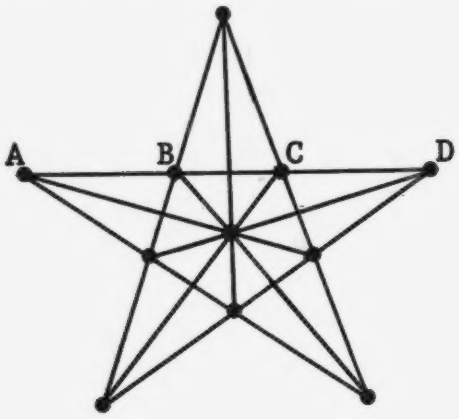
13. Polytopes with rational vertices.

We now turn to a topic in integer programming which extends some of the ideas we discussed after Section 10. I first heard of the following problem from Klee. For any d-polytope P, does there exist a polytope P' of the same combinatorial type which has all of its vertices at rational points in E^d? Equivalently, we could ask, could all the coordinates of each vertex be integers? Until a few years ago there was nothing in the literature which answers this. The answer to the question is clearly "yes" in E^2, is "yes" but not obviously so in E^3, is unknown in dimensions 4 through 7, and is "no" in E^8. If we start with a rational simplex (rational here means that the vertices have rational coordinates) in E^3 and continue to transform it by the elementary transformations of Steinitz's theorem as in Corollary 12.3 above, then it is no loss of generality to assume that each transformed polytope is again rational. Thus the problem in E^3 can be solved using the proof of Steinitz's theorem. In E^d it is always possible to get a rational simplex, or a rational realization of any simplicial polytope, but in general "moving a vertex a little" might change the combinatorial type of P even if $\rho(P,P') \leqq \epsilon$ in the Hausdorff metric.

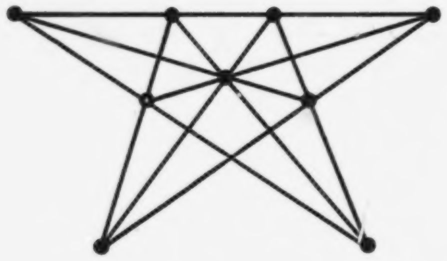
Perles (see Grünbaum [1], Section 5.5) has given an example of a (convex) 8-polytope with twelve vertices which has no rational realization. This is the best example possible in E^8 in the sense that if $P \subset E^d$ has at most $d+3$ vertices, then it has a rational realization. The irrationalities of Perles' example are not too bad in the sense that we could realize all coordinates of its vertices in the field which is the simple extension of the rationals by $\sqrt{5}$. More generally, it is possible to obtain realizations of all d-polytopes using the field of real algebraic numbers. Even in E^8, though, it is not known whether every polytope may be realized using a field which is an extension of the rationals by a finite number of irrationals.

Our next topic will show that this type of occurrence is not too surprising. We will consider arrangements of lines in the projective plane (or arrangements of hyperplanes in projective d-space). By an arrangement we mean a set of lines in the projective plane which do not form a pencil at one point. We are interested in questions like, what are the regions into which the plane is divided, how many vertices are there, etc. We can ask when any two arrangements are combinatorially equivalent in the sense that the equivalence preserves the number of lines and vertices and their incidences. In particular we can ask if every arrangement has an equivalent rational arrangement; i.e., an arrangement where the coordinates of all vertices are rational.

In answer to the last question, the arrangement of ten lines and eleven vertices as shown cannot have a rational realization. It can be shown that any arrangement combinatorially equivalent to this example is already projectively equivalent to it. Hence the cross-ratio $(A,B;C,D)$ is irrational in each realization. The example is actually larger than necessary.



The subarrangement of nine points and nine lines as shown has the same property, and is best possible in the sense that any arrangement in E^2 with at most eight points or eight lines has a rational realization.



Perles' example in E^8, referred to above, makes use of this last example. His argument uses a particular transformation which maps a d-dimensional arrangement with v vertices into (v-d-1)-dimensional space. We have used the word "arrangement" here rather than "configuration," since a configuration is usually considered to have a constant number of points on each line.

Discussion.

It was pointed out during a discussion that the two examples in E^2 which do not have rational realizations, can in fact be realized in the 2-dimensional vector space over the field $R(\sqrt{5})$ (i.e., the simple extension of the rationals by $\sqrt{5}$). On the other hand, they cannot be realized if the field is $R(\sqrt{2})$.

The figure with ten lines and eleven vertices is "self-conjugate" in the sense that any combinatorially equivalent arrangement is either essentially the same or is one where the five vertices on the points exchange places with the five vertices on the inner pentagon.

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